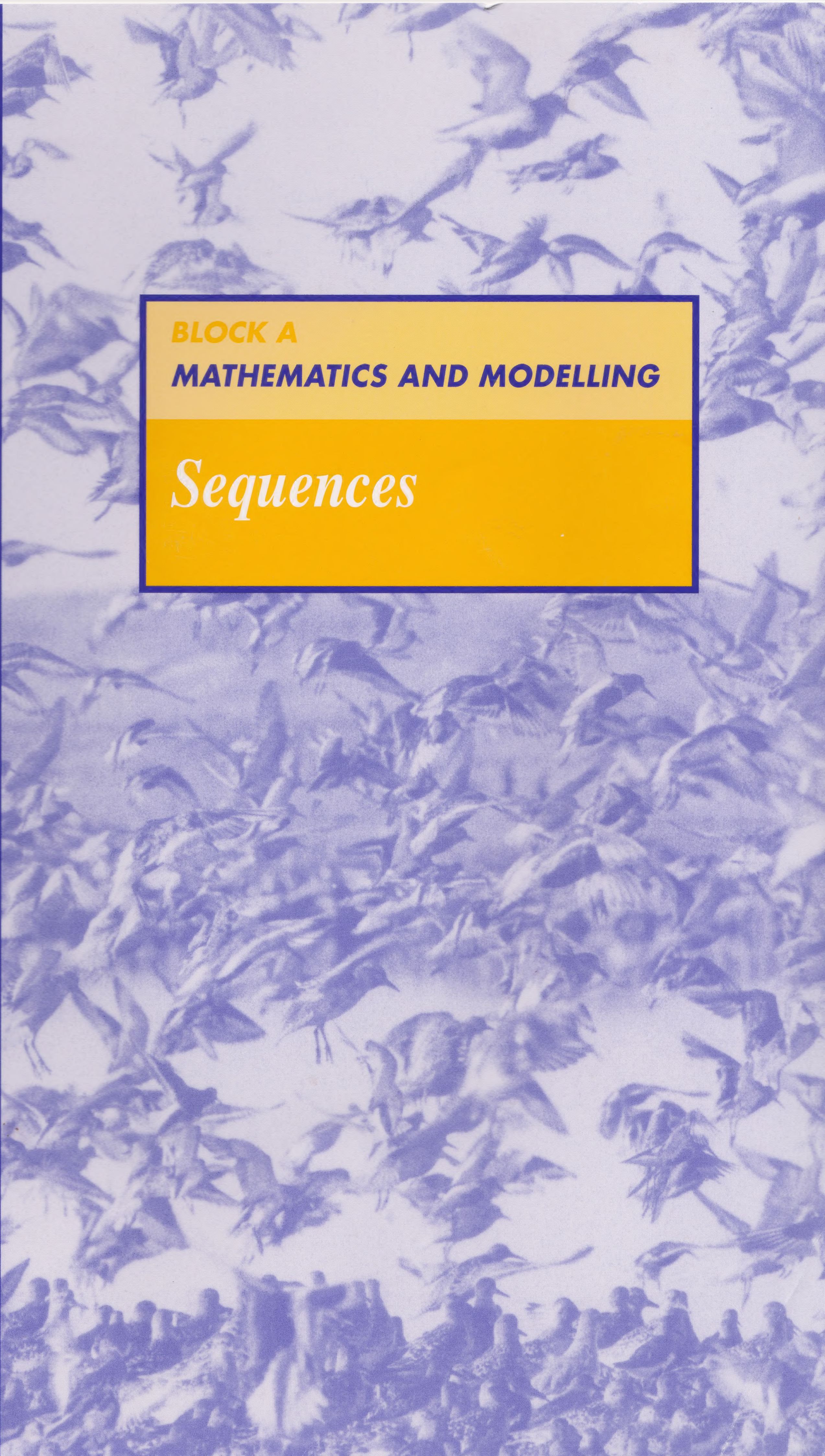
MST121 Chapter A1

The Open University

A first level Interdisciplinary course

Using Mathematics



BLOCK A

MATHEMATICS AND MODELLING

Sequences

CHAPTER

A1

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Prepared by the course team

CHAPTER

A1

About this course

This course, MST121 *Using Mathematics*, and the courses MU120 *Open Mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MST121 uses the software program Mathcad (MathSoft, Inc.) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of the course.

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Study guide

There are seven sections to this chapter, which are intended to be studied consecutively. The first six sections should take one and a half to three hours of study each, the longest one being Section 4, which includes an audio tape. Section 6 contains only computer-based work. Section 7 is shorter and includes *no* mathematical problems.

All sections require the use of this main text; Subsection 4.2 requires the use of an audio cassette player, whereas Section 6 requires the use of the computer and Computer Book A.

The pattern of study for each session might be as follows.

Study session 1: Sections 1 and 2.

Study session 2: Section 3.

Study session 3: Section 4.

Study session 4: Sections 5.

Study session 5: Sections 6 and 7.

If it is more convenient for you, Section 7 may be studied before Section 6.

Before studying this chapter, you should be familiar with the following topics, which are covered in the *Revision Pack* and Chapter A0:

◇ solving pairs of simultaneous equations;

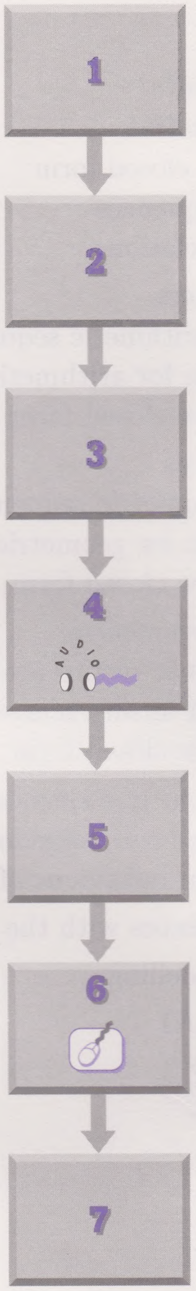
◇ rounding numbers;

◇ sketching simple graphs;

◇ percentages;

◇ basic algebraic manipulation.

> The optional Video Band A(ii) *Algebra workout – Simultaneous equations* could be viewed at any stage during your study of this chapter.



Introduction

This chapter introduces the concept of a *sequence*, which is the mathematical name for a list of numbers arranged in a particular order. Below, six of the sequences discussed in the chapter are given. They arise from situations as diverse as estimating the height of a bouncing ball, predicting deer populations and calculating mortgage repayments.

For convenience, just the first four *terms* of each sequence are given.

$5,\ 8,\ 11,\ 14,\ \ldots$

$1000,\ 950,\ 900,\ 850,\ \ldots$

$1000, 1050, 1102.50, 1157.63,\ \ldots$

$2000,\ 1400,\ 980,\ 686,\ \ldots$

$6000,\ 6400,\ 6860,\ 7389,\ \ldots$

$10\,000,\ 9697.57,\ 9380.02,\ 9046.59,\ \ldots$

Each of these sequences has a definite mathematical pattern, which you will learn to identify. You will also learn how to describe the long-term behaviour of the numbers in such sequences. This is of importance for understanding sequences that arise in real-world contexts.

You may have spotted the patterns in some of these sequences.

Sequences are introduced in Section 1 using two different approaches, known as *closed forms* and *recurrence systems*. Both of these methods of specifying sequences will appear frequently in the course, in particular in computer work. In Sections 2, 3 and 4, you will study three special types of sequences which include the six examples above.

Guidance on a possible structure for your study is provided on the opposite page.

In Sections 5 and 6, you will investigate such sequences, by hand and with the aid of the computer. This leads to an appreciation of how sequences behave in the long term, when we consider a large number of terms.

Finally, in Section 7, we discuss briefly the framework within which the course will apply mathematics to try to solve real-world problems; the process we use when dealing with such applications is called *mathematical modelling*.

1 What is a sequence?

1.1 Sequence notation

We frequently meet lists of numbers arranged in order. For example, a list of numbers may be used to represent a quantity that is varying over time, such as the midday temperature (in °C) recorded each day for a week:

$$13, 12, 10, 10, 10, 9, 8,$$

or the amount of money (in £) in a savings account on each 1 January over a five-year period:

$$1000, 1050, 1102.50, 1157.63, 1215.51.$$

You met the names for various types of numbers in Chapter A0, Section 3.

Lists that relate to real-world quantities are usually **finite**; that is, the list has a first number and a final number. However, the list of odd natural numbers

$$1, 3, 5, 7, 9, \ldots$$

goes on forever; such a list is called **infinite**.

In mathematics, the word **sequence** is used, rather than 'ordered list', and the numbers in the list are called the **terms** of the sequence. For example,

$$13, 12, 10, 10, 10, 9, 8$$

is a finite sequence, whose first term is 13 and whose final term is 8. On the other hand, the infinite sequence 1, 3, 5, 7, ... has first term 1 but no final term.

Many sequences have a structure, or pattern, which allows us to give a concise description of the sequence and which helps us to understand the sequence's behaviour. For example, the sequence 1, 3, 5, 7, ... has a very simple mathematical structure; each term is exactly 2 more than the previous term:

$$3 = 1 + 2,\ 5 = 3 + 2,\ 7 = 5 + 2,\ \ldots.$$

The savings account sequence 1000, 1050, ... also has a simple structure, though this is less obvious. For this sequence, it can be checked that, to three significant figures:

$$\frac{1050}{1000} = 1.05,\quad \frac{1102.50}{1050} = 1.05,\quad \frac{1157.63}{1102.50} = 1.05,\quad \frac{1215.51}{1157.63} = 1.05$$

so each term is 1.05 times the previous term (approximately at least).

See Section 3 for more details of this sequence.

This pattern arises because this savings account pays compound interest of 5% per annum, and there have been no withdrawals.

We would not expect the above temperature sequence to have any simple mathematical structure; indeed, if you were to discover such a structure in a sequence of temperatures, then you would be very suspicious. However, it may be noted that the temperatures in the above sequence decrease from day to day, and there may be some *meteorological* reason for this.

In this chapter, we investigate various types of sequence which *do* have a simple underlying mathematical structure. Such sequences arise in various real-world situations and have many applications.

First, however, here is some notation associated with sequences. In elementary algebra, we use letters such as $a, b, c, x, y, z, A, B, C$, and so on, to represent variables. With a sequence, we represent the terms by using one particular letter with an attached subscript; this subscript is an integer that indicates which term of the sequence is meant. Thus the sequence

$$a_1, a_2, a_3, \ldots, a_{10}$$

has 10 terms, the first being a_1 and the final one being a_{10}. This is called **subscript** notation (or **suffix** notation). Sometimes it is possible to choose an appropriate letter for a sequence. For example, we might use t for the temperature sequence. Since the first term is 13, it is natural to write $t_1 = 13$, and so on, giving

These terms are read as: a-one (or a-sub-one), a-two (or a-sub-two), and so on.

$$t_1 = 13,\ t_2 = 12,\ t_3 = 10,\ t_4 = 10,\ t_5 = 10,\ t_6 = 9,\ t_7 = 8.$$

The use of appropriate letters can be helpful, especially when dealing with several sequences, but it is not necessary and indeed not always possible.

We can use upper- or lower-case letters to represent sequences.

If we want to refer to a general term of a sequence without committing ourselves to a particular term, then we use a letter for the subscript as well, as follows:

a_n denotes the term of the sequence with subscript n.

Here n represents a natural number in the appropriate range; for example, for the sequence $a_1, a_2, \ldots, a_{10}$, the range of n is $1, 2, \ldots, 10$. In mathematics, the letter n usually represents a natural number or, more generally, an integer – that is, a number of the form $\ldots, -2, -1, 0, 1, 2, \ldots$. Note that we shall also use the name 'a_n' to refer to the *sequence* whose general term is a_n. (It will be clear from the context which meaning is appropriate.)

Other letters commonly used to represent integers are i, j, k, l, m, p and q.

Some texts use notation such as $\{a_n\}, (a_n)$, or $\langle a_n \rangle$ to distinguish a sequence from its general term.

Activity 1.1 General terms

For the temperature sequence above, write down the value of t_n for each of the following values of n.

(a) $n = 2$ (b) $n = 6$

Solutions are given on page 41.

1.2 Sequences in closed form

The notation a_n is useful when we want to specify, or define, a sequence by giving a formula for the terms. For example, suppose that we wish to specify the infinite sequence of **perfect squares**:

$$1, 4, 9, 16, 25, \ldots.$$

The terms of this sequence are the squares of the natural numbers $1, 2, 3, 4, 5, \ldots$; that is,

$$1 = 1^2,\ 4 = 2^2,\ 9 = 3^2,\ 16 = 4^2,\ 25 = 5^2, \ldots.$$

If we choose to represent this sequence using the letter s (for square), then

$$s_1 = 1^2 = 1,\ s_2 = 2^2 = 4,\ s_3 = 3^2 = 9,\ s_4 = 4^2 = 16, \ldots.$$

For a general natural number n, we have $s_n = n^2$ and this is a formula for the general term or nth term of the sequence. To complete the specification of the sequence, we need to state the range of values of the subscript n. This we shall set out using brackets, as follows:

An alternative setting out, not employing brackets, is

$$s_n = n^2, \quad n = 1, 2, 3, \ldots.$$

$$s_n = n^2 \quad (n = 1, 2, 3, \ldots).$$

The word 'closed' contrasts this way of defining a sequence with other ways that you will meet later.

Such a formula for defining a sequence in terms of the subscript n is called a **closed form** (or a **closed-form formula**). A closed form for a sequence enables us to calculate any term of the sequence directly once we are given the value of n. Unfortunately, however, not all sequences have such a formula. The existence of a closed form provides a convenient way of referring to a sequence. For example, we might say 'the sequence n^2'. This form of reference will be useful in Sections 5 and 6.

Activity 1.2 Using closed forms

Write down the first five terms, and the 100th term, of each of the following sequences.

(a) $a_n = 7n \quad (n = 1, 2, 3, \ldots)$

(b) $b_n = 1/n \quad (n = 1, 2, 3, \ldots)$

(c) $c_n = (-1)^{n+1} \quad (n = 1, 2, 3, \ldots)$

Solutions are given on page 41.

Comment

The sequence in part (c) is often used when we need to specify a sequence whose terms alternate in sign.

Activity 1.3 Spotting a closed form

(a) Specify a sequence in closed form whose first four terms are

$$1, 8, 27, 64.$$

(*Hint*: Try to find a common property of these four numbers.)

(b) Use your closed form to write down the 10th term in your sequence.

Solutions are given on page 41.

For all the sequences we have considered so far, the first term of the sequence has subscript 1; that is, they are of the form $a_1, a_2, a_3, \ldots$. This seems natural and easy to remember, but there are occasions when it is more convenient to be flexible about how we represent the first term. Consider, for example, the following sequence:

$$1, 2, 4, 8, 16, \ldots,$$

which arises in many contexts. This is the sequence of 'powers of 2':

Remember that $a^0 = 1$, for all non-zero values of a; that is, $a^0 = 1$, for $a \neq 0$.

$$2^0, 2^1, 2^2, 2^3, 2^4, \ldots.$$

We can define this sequence in closed form using $a_n = 2^n$, but only if we allow the subscript 0:

$$a_n = 2^n \quad (n = 0, 1, 2, \ldots).$$

In this case, the simplicity of the formula $a_n = 2^n$ generally outweighs the slight oddity of starting with a_0. There is the possibility of confusion in

having a_0 as the first term of a sequence, with a_1 as second term, and so on, but with practice this should not cause difficulties. If it were essential to have first term a_1, then we could define the 'powers of 2' sequence as follows:

$$a_n = 2^{n-1} \quad (n = 1, 2, 3, \ldots).$$

For example,

$$a_1 = 2^{1-1} = 2^0 = 1.$$

Further flexibility about the first term of a sequence is sometimes useful, as in Activity 1.4(b). To avoid constant repetition, we adopt the following convention.

Convention

When we discuss a sequence a_n, we assume that the first term is a_1 unless we indicate otherwise.

Activity 1.4 Unusual first terms

Write down the first three terms of each of the following sequences.

(a) $a_n = 3^n \quad (n = 0, 1, 2, \ldots)$

(b) $b_n = \dfrac{1}{n(n-1)} \quad (n = 2, 3, 4, \ldots)$

(c) $c_n = \dfrac{1}{(n+1)n} \quad (n = 1, 2, 3, \ldots)$

Solutions are given on page 41.

Comment

The sequences b_n and c_n illustrate the fact that two sequences which appear to be different at first sight can actually have exactly the same terms.

1.3 Graphs of sequences

The first few terms of a given sequence often suggest the sequence's mathematical structure, especially if the terms are simple and familiar. For example, a sequence whose first five terms are

$$7, 14, 21, 28, 35$$

seems very likely to be successive multiples of 7, with the closed form in Activity 1.2(a). On the other hand, the savings account sequence

$$1000, 1050, 1102.50, 1157.63, 1215.51$$

does not immediately appear to have such a structure. One way to detect the presence of structure in a given sequence a_n is to plot a graph of the sequence. It is convenient to use the horizontal axis for the subscript n and the vertical axis for the terms a_n, representing each point on the graph by a symbol such as a cross. For example, the graph of the above savings account sequence is as follows.

Sometimes we join up the points to indicate a pattern that is present.

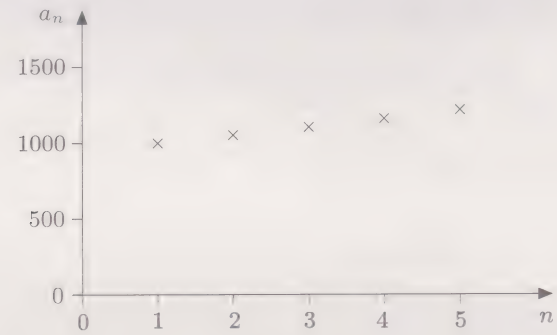


Figure 1.1 Savings account sequence

This graph shows that the terms of this sequence increase from year to year in a regular manner. This suggests the possibility of a simple mathematical structure, but it does not immediately help us to find this structure.

Activity 1.5 Plotting graphs

Plot a graph for each of the sequences given in Activity 1.2, showing the first five terms in each case.

Solutions are given on page 41.

1.4 Recurrence relations

The two sequences $a_n = 7n$ $(n = 1, 2, 3, \ldots)$ with terms

$$7, 14, 21, 28, 35, \ldots,$$

and $b_n = 2^n$ $(n = 0, 1, 2, \ldots)$ with terms

$$1, 2, 4, 8, 16, \ldots,$$

have a similar property. In each case, the *next term* of the sequence can be found from the *current term* of the sequence by using a formula. For the sequence a_n the next term is always 7 more than the current term:

$$a_2 = a_1 + 7, a_3 = a_2 + 7, a_4 = a_3 + 7, \ldots,$$

whereas for the sequence b_n the next term is always twice the current term:

$$b_1 = 2b_0, b_2 = 2b_1, b_3 = 2b_2, \ldots.$$

If the current term has subscript n, then the next term has subscript $n + 1$, and so we can write

$$a_{n+1} = a_n + 7 \quad (n = 1, 2, 3, \ldots)$$

and

$$b_{n+1} = 2b_n \quad (n = 0, 1, 2, \ldots).$$

Such a formula, which allows one term of a sequence to be obtained from the previous term, is called a **recurrence relation**, because the same relationship between adjacent terms keeps recurring. It is called a **first-order** recurrence relation because only one previous term is involved.

If we know a recurrence relation for a sequence, and also know the first term of the sequence, then in principle we can determine any required term of the sequence by starting from the first term and repeatedly applying the recurrence relation. For example, from the recurrence relation

$$x_{n+1} = x_n^2 \quad (n = 1, 2, 3, \ldots),$$

and the first term $x_1 = 2$, we can successively calculate

$$x_1 = 2,$$

$$x_2 = x_1^2 = 2^2 = 4,$$

$$x_3 = x_2^2 = 4^2 = 16,$$

$$x_4 = x_3^2 = 16^2 = 256.$$

and so on. Notice that if we keep the same recurrence relation but change the first term x_1, then a different sequence is obtained; for example, try $x_1 = 1$, with the above recurrence relation.

Taken together, the specification of a first term, a recurrence relation and a subscript range is called a **recurrence system** and the resulting sequence is called a **recurrence sequence**. We shall display the three parts of such a recurrence system as follows:

$$x_1 = 2, \quad x_{n+1} = x_n^2 \quad (n = 1, 2, 3, \ldots),$$

with the first term of the sequence on the left and the recurrence relation, including the range of values of n, on the right.

If the first term of this sequence is defined not as x_1 but as x_0, say, then the range of values of n will have to begin with 0 rather than 1.

The next activity gives you some practice at finding the first few terms of several sequences defined by recurrence systems.

Activity 1.6 Using recurrence systems

Write down the first five terms of each of the following sequences.

(a) $a_1 = 0, \quad a_{n+1} = 2a_n + 1 \quad (n = 1, 2, 3, \ldots)$

(b) $b_1 = 1, \quad b_{n+1} = b_n^2 - 1 \quad (n = 1, 2, 3, \ldots)$

(c) $c_0 = 2, \quad c_{n+1} = \frac{1}{2}(c_n + 2/c_n) \quad (n = 0, 1, 2, \ldots)$

In part (c), round your answers to six decimal places where appropriate.

Solutions are given on page 41.

Comment

The recurrence sequence in part (c) very rapidly gives good approximations to $\sqrt{2}$, as you can check by finding $\sqrt{2}$ on your calculator.

Note that a recurrence relation may express x_n in terms of x_{n-1}, rather than x_{n+1} in terms of x_n. If this is the case, then the range of values of n will need adjustment. For example, the sequence in Activity 1.6(a) can also be specified as follows:

$$a_1 = 0, \quad a_n = 2a_{n-1} + 1 \quad (n = 2, 3, 4, \ldots).$$

In the next three sections we study several types of recurrence systems, and investigate whether a closed form can be found for the corresponding sequences. Many sequences defined by recurrence systems do have closed forms, but not all.

Summary of Section 1

This section has introduced:

- the notation a_n for a finite or infinite sequence, and for its term with subscript n;
- a convention about the first term of a sequence;
- specifying a sequence using a closed form;
- sketching the graph of a sequence;
- specifying a sequence using a recurrence system.

Exercises for Section 1

Exercise 1.1

Write down the first five terms and the 10th term of each of the following sequences.

(a) $a_n = 2^n - n^2 \quad (n = 1, 2, 3, \ldots)$

(b) $b_n = 1/n^2 \quad (n = 1, 2, 3, \ldots)$

(c) $c_n = \sqrt{n} \quad (n = 0, 1, 2, \ldots)$

Exercise 1.2

Plot a graph for each of the sequences given in Exercise 1.1, showing the first five terms in each case.

Exercise 1.3

Write down the first five terms of each of the following sequences.

(a) $a_1 = 1, \quad a_{n+1} = \dfrac{1}{a_n + 1} \quad (n = 1, 2, 3, \ldots)$

(b) $b_0 = 1, \quad b_{n+1} = 2^{b_n} \quad (n = 0, 1, 2, \ldots)$

(c) $c_1 = 2, \quad c_{n+1} = \dfrac{1}{c_n} \quad (n = 1, 2, 3, \ldots)$

2 Arithmetic sequences

In this section, we consider sequences defined by a rather basic type of recurrence relation which occurs frequently in practice.

2.1 What is an arithmetic sequence?

Pronounce 'arithmetic' here with emphasis on the syllable 'met'.

We begin with two sequences which arise in different ways but which are of a similar *mathematical* type.

First, consider the finite sequence

$$5, 8, 11, 14, \ldots, 38.$$

This represents the total number of books that have been received by the end of successive months by a member of a book club. The club sends five books in January and three books in each month after that, for the rest of the year. We shall call this sequence b_n, so that $b_1 = 5$, $b_2 = 8$, $b_3 = 11$, and so on.

To get from any term in this sequence to the next, we *add* the same number each time:

$$8 = 5 + 3,$$

$$11 = 8 + 3,$$

$$14 = 11 + 3,$$

and so on. The number 3 occurs here because it is the number of books received each month after the first. Thus this sequence can be defined by the recurrence system

$$b_1 = 5, \quad b_{n+1} = b_n + 3 \quad (n = 1, 2, 3, \ldots, 11).$$

The last value in the range of n is 11 because the subject of the recurrence relation is b_{n+1} and the final term in the sequence is b_{12}.

Next, consider the finite sequence

$$1000, 950, 900, 850, \ldots, 0.$$

This represents the volume, measured in litres on successive Saturdays, of oil in a tank supplying a boiler that is assumed to use 50 litres of oil per week. We shall call this sequence v_n, so that $v_1 = 1000$, $v_2 = 950$, $v_3 = 900$, and so on.

Once again, to get from any term in this sequence to the next, we *add* the same number each time:

$$950 = 1000 + (-50),$$

$$900 = 950 + (-50),$$

$$850 = 900 + (-50),$$

and so on. The negative number -50 occurs here because each week the volume in the tank is *reduced* by 50 litres. Thus this sequence can be defined by the recurrence system

Subtracting 50 is the same as adding -50.

$$v_1 = 1000, \quad v_{n+1} = v_n - 50 \quad (n = 1, 2, 3, \ldots, 20).$$

The last value in the range of n is 20 because 1000 litres at 50 litres per week lasts 20 weeks, so that the final term in the sequence is $v_{21} = 0$.

The name arises because each term (other than the first) is the *arithmetic mean*, or average, of its two neighbouring terms.

Any sequence with this structure – the addition of a fixed number to obtain the next term – is called an **arithmetic sequence** (also commonly called an **arithmetic progression**). Thus a general arithmetic sequence is given by:

$$x_1 = a, \quad x_{n+1} = x_n + d \quad (n = 1, 2, 3, \ldots),$$

where a is the first term and d is the constant difference $x_{n+1} - x_n$ between any pair of successive terms, often called the **common difference**. Choosing the values of a and d determines a particular arithmetic sequence; we call a and d the **parameters** of the arithmetic sequence. For example, the book sequence has $a = 5$ and $d = 3$, and the oil sequence has $a = 1000$ and $d = -50$. There are minor variations of this definition; for example, the range of values of the subscript n may be finite, as in the book sequence and the oil sequence, or the first term may be x_0:

For any particular arithmetic sequence, the parameters a and d have fixed values, so we could have called a and d 'constants'.

$$x_0 = a, \quad x_{n+1} = x_n + d \quad (n = 0, 1, 2, \ldots).$$

Activity 2.1 Recognising arithmetic sequences

Which of the following sequences is an arithmetic sequence? For each arithmetic sequence, write down the values of the parameters a and d.

(a) $x_1 = -1, \quad x_{n+1} = x_n + 1 \quad (n = 1, 2, 3, \ldots)$

(b) $y_1 = 2, \quad y_{n+1} = -y_n + 1 \quad (n = 1, 2, 3, \ldots)$

(c) $z_0 = 1, \quad z_{n+1} = z_n - 1.5 \quad (n = 0, 1, 2, \ldots)$

Solutions are given on page 42.

Suppose now that you have been given the first few terms of an arithmetic sequence x_n. How can you find its parameters a and d? Well, a is just the first term, and d is the difference $x_{n+1} - x_n$ between any pair of adjacent terms x_{n+1} and x_n in the sequence.

For example,

$$d = x_2 - x_1,$$
$$d = x_3 - x_2.$$

In the next activity you are asked to write down the recurrence system for some arithmetic sequences.

Activity 2.2 Finding parameters for arithmetic sequences

Each of the three sequences in this activity is arithmetic and infinite.

By our convention, each of these sequences should start with a term with subscript 1.

(a) For the sequence x_n, whose first four terms are $1, 4, 7, 10$,

(i) find the values of a and d, and write down the corresponding recurrence system;

(ii) calculate the next two terms, and plot a graph of the first six terms.

(b) Repeat part (a) for the sequence y_n, whose first four terms are $2.1, 3.2, 4.3, 5.4$.

(c) Repeat part (a) for the sequence z_n, whose first four terms are $1, 0.9, 0.8, 0.7$.

Do the graphs you have plotted have any obvious features in common?

Solutions are given on page 42.

2.2 A closed form for arithmetic sequences

Arithmetic sequences have a particularly simple form: to get from one term to the next, we add on the same number each time. This pattern allows us to obtain a closed form for such sequences, which makes them easier to handle mathematically.

To illustrate how this is done, we consider the book sequence:

$$5, 8, 11, 14, \ldots, 38.$$

This is an arithmetic sequence with parameters $a = 5$ and $d = 3$, which may be defined as

$$b_1 = 5, \quad b_{n+1} = b_n + 3 \quad (n = 1, 2, 3, \ldots, 11),$$

so that $b_1 = 5$, $b_2 = 8$, and so on. The way in which the terms of this sequence are obtained from the recurrence relation can be pictured as follows.

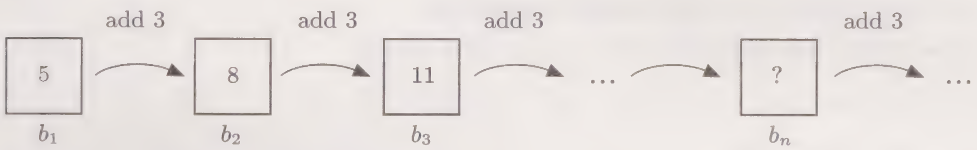


Figure 2.1 Obtaining the terms of the book sequence

Starting from $b_1 = 5$,

to obtain b_2, we add 3,

to obtain b_3, we add 3 twice,

to obtain b_4, we add 3 three times,

and so on. In each case the number of added 3s is one fewer than the subscript on the left. Thus, to obtain the general term b_n, we have to add 3 exactly $n - 1$ times; that is, we add $3(n - 1)$. This gives the value of the general term as:

$$b_n = 5 + 3(n - 1) = 3n + 2 \quad (n = 1, 2, 3, \ldots, 12).$$

Note that the last value in this range of n is 12.

For example, using this closed form we find that $b_4 = 3 \times 4 + 2 = 14$, as expected.

Activity 2.3 Obtaining closed forms for arithmetic sequences

Use similar reasoning to obtain a closed form for the oil sequence:

$$v_1 = 1000, \quad v_{n+1} = v_n - 50 \quad (n = 1, 2, 3, \ldots, 20).$$

Check that your closed form gives the correct value for v_4.

A solution is given on page 42.

The same reasoning can be applied to a general arithmetic sequence

$$x_1 = a, \quad x_{n+1} = x_n + d \quad (n = 1, 2, 3, \ldots).$$

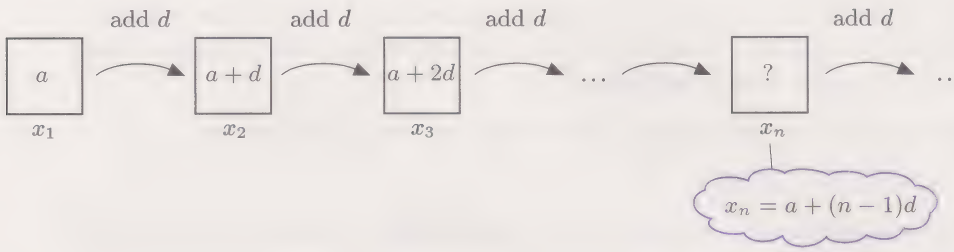


Figure 2.2 Obtaining the terms of a general arithmetic sequence

To obtain the general term x_n, we start with $x_1 = a$ and add d exactly $n - 1$ times, so $x_n = a + (n-1)d$. This result is displayed below.

Closed form for arithmetic sequences

The arithmetic sequence with recurrence system

$$x_1 = a, \quad x_{n+1} = x_n + d \quad (n = 1, 2, 3, \ldots)$$

has the closed form

$$x_n = a + (n-1)d \quad (n = 1, 2, 3, \ldots).$$

The expression $a + (n-1)d$ may be simplified when a and d have particular values. For example,

$$5 + 3(n-1) = 3n + 2.$$

Whenever a new formula has been obtained, it is good practice to check that the formula gives the correct answer in simple cases. This checking does not guarantee that the formula is correct, but it may uncover errors in the derivation. For example, when $n = 1$ and $n = 2$, this formula gives the correct values $x_1 = a$ and $x_2 = a + d$. Also, if $d = 0$, then the closed form gives the sequence $x_n = a$ $(n = 1, 2, 3, \ldots)$, as expected. The latter sequence is a **constant sequence**, in which each term has the same value.

The next activity provides the opportunity for you to apply this closed form to particular sequences.

Activity 2.4 Writing down closed forms

Write down a closed form for each of the sequences in Activity 2.2. In each case, check that your answer gives the correct value for the fourth term, and also calculate the 100th term of each sequence.

Solutions are given on page 42.

2.3 An alternative closed form

As mentioned in Section 1, it is not always convenient for the first term of a sequence to have subscript 1. For example, we often consider sequences with first term x_0. An arithmetic sequence of this type is specified by

$$x_0 = a, \quad x_{n+1} = x_n + d \quad (n = 0, 1, 2, \ldots).$$

In order to obtain x_n, we start with $x_0 = a$ and have to add on d exactly n times. This gives the closed form

$$x_n = a + nd \quad (n = 0, 1, 2, \ldots).$$

For example,

$$x_1 = a + d,$$
$$x_2 = a + 2d.$$

This version of the closed form is slightly neater than the one with first term x_1, and we sometimes prefer it for this reason. Remember, however, that for such a sequence care is needed in identifying particular terms; for example, x_{10} is the *eleventh* term of the sequence.

Activity 2.5 Using the alternative closed form

Write down a closed form for each of the following arithmetic sequences, and hence calculate the 100th term of each sequence.

(a) $x_0 = 1, \quad x_{n+1} = x_n + 2 \quad (n = 0, 1, 2, \ldots)$

(b) $y_0 = 10, \quad y_{n+1} = y_n - 0.01 \quad (n = 0, 1, 2, \ldots)$

Solutions are given on page 43.

Summary of Section 2

This section has introduced:

◇ arithmetic sequences defined by recurrence systems of the form

$$x_1 = a, \quad x_{n+1} = x_n + d \quad (n = 1, 2, 3, \ldots),$$

which have closed form

$$x_n = a + (n-1)d \quad (n = 1, 2, 3, \ldots);$$

◇ arithmetic sequences defined by recurrence systems of the form

$$x_0 = a, \quad x_{n+1} = x_n + d \quad (n = 0, 1, 2, \ldots),$$

which have closed form

$$x_n = a + nd \quad (n = 0, 1, 2, \ldots);$$

◇ a method for finding the parameters a and d of a given arithmetic sequence.

Exercises for Section 2

Exercise 2.1

Each of the sequences in this exercise is arithmetic and infinite.

(a) For the sequence x_n, whose first four terms are 100, 95, 90, 85,

(i) find the values of the parameters a and d, and write down the corresponding recurrence system;

(ii) calculate the next two terms and plot a graph of the first six terms.

(b) Repeat part (a) for the sequence y_n, whose first four terms are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Exercise 2.2

Write down a closed form for each of the following sequences, and hence calculate the 10th term of each sequence.

(a) $x_1 = 2, \quad x_{n+1} = x_n - 3 \quad (n = 1, 2, 3, \ldots)$

(b) $y_0 = 0, \quad y_{n+1} = y_n + 0.2 \quad (n = 0, 1, 2, \ldots)$

3 Geometric sequences

In this section, we consider sequences defined by another basic type of recurrence relation that occurs frequently in practice.

3.1 What is a geometric sequence?

We begin once again with two sequences from different real-world contexts.

See Section 1.

First, consider the savings account sequence

$$1000, 1050, 1102.50, 1157.63, 1215.51.$$

This represents the amount of money (in £) in a savings account on each successive 1 January over a five-year period. We shall call this sequence s_n, so that $s_1 = 1000$, $s_2 = 1050$, $s_3 = 1102.50$, and so on.

To get from any term in this sequence to the next, we *multiply* by the same number each time:

$$1050 = 1.05 \times 1000,$$

$$1102.50 = 1.05 \times 1050,$$

$$1157.63 = 1.05 \times 1102.50,$$

Note that 1157.63 is 1157.625 rounded to two decimal places – that is, to the nearest penny.

and so on. The number 1.05 occurs here because the interest added on at the end of each year is 0.05 times (that is, 5% of) the amount in the account at the *beginning* of the year. Thus this sequence can be defined by the recurrence system

$$s_1 = 1000, \quad s_{n+1} = 1.05 s_n \quad (n = 1, 2, 3, 4),$$

since the final term in this finite sequence is s_5.

Next, consider the sequence

$$2000, 1400, 980, 686, 480.2, \ldots.$$

This represents the height, measured in millimetres, of successive bounces of a ball which is assumed to rebound to 70% of the height from which it fell. We shall call this sequence h_n, so that $h_1 = 2000$, $h_2 = 1400$, $h_3 = 980$, and so on.

This representation of a bouncing ball is considered further in Section 7.

Once again, to get from any term in this sequence to the next, we *multiply* by the same number each time:

$$1400 = 0.7 \times 2000,$$

$$980 = 0.7 \times 1400,$$

$$686 = 0.7 \times 980,$$

and so on. The number 0.7 occurs here because each successive height is 70% of the previous one. Thus this sequence can be defined by the recurrence system

$$h_1 = 2000, \quad h_{n+1} = 0.7 h_n \quad (n = 1, 2, 3, \ldots),$$

where we have assumed (unrealistically) that the ball continues to bounce indefinitely.

Any sequence with this structure – multiplication by a fixed number to obtain the next term – is called a **geometric sequence** (also commonly called a **geometric progression**). Thus a general geometric sequence is given by:

$$x_1 = a, \quad x_{n+1} = rx_n \quad (n = 1, 2, 3, \ldots),$$

where a is the first term and r is the constant ratio x_{n+1}/x_n of any two successive terms, often called the **common ratio**. Choosing the values of a and r determines a particular geometric sequence; we call a and r the **parameters** of the geometric sequence. For example, the savings account sequence has $a = 1000$ and $r = 1.05$, and the bouncing ball sequence has $a = 2000$ and $r = 0.7$.

The name arises because, for a geometric sequence in which all terms are positive, each term (other than the first) is the *geometric mean* of its two neighbouring terms – that is, the square root of their product.

Once again, there are minor variations; for example, the range of values of the subscript n may be finite, as in the savings account sequence, or the first term may be x_0:

$$x_0 = a, \quad x_{n+1} = rx_n \quad (n = 0, 1, 2, \ldots).$$

Activity 3.1 Recognising geometric sequences

Which of the following sequences is a geometric sequence? For each geometric sequence, write down the values of a and r.

(a) $x_1 = -1, \quad x_{n+1} = 3x_n \quad (n = 1, 2, 3, \ldots)$

(b) $y_0 = 1, \quad y_{n+1} = -0.9y_n \quad (n = 0, 1, 2, \ldots)$

(c) $z_1 = 2, \quad z_{n+1} = -z_n + 1 \quad (n = 1, 2, 3, \ldots)$

Solutions are given on page 43.

Suppose now that you know the first few terms of a geometric sequence x_n and you want to find the recurrence system that generates it. As with arithmetic sequences, a is just the first term. But r is the ratio x_{n+1}/x_n of any pair of adjacent terms x_{n+1} and x_n.

For example,

$$r = x_2/x_1,$$
$$r = x_3/x_2.$$

In the next activity, you are asked to write down the recurrence system for some geometric sequences.

Activity 3.2 Finding parameters for geometric sequences

Each of the three sequences in this activity is geometric and infinite.

(a) For the sequence x_n, whose first four terms are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$,

(i) find the values of a and r, and write down the corresponding recurrence system;

(ii) calculate the next two terms, and plot a graph of the first six terms.

(b) Repeat part (a) for the sequence y_n, whose first four terms are $4.2, 7.14, 12.138, 20.6346$.

(c) Repeat part (a) for the sequence z_n, whose first four terms are $2, -2, 2, -2$.

Do the graphs you have plotted have any obvious features in common?

Solutions are given on page 43.

3.2 A closed form for geometric sequences

Geometric sequences, like arithmetic sequences, have a particularly simple form: to get from one term to the next, we multiply by the same number each time. This pattern allows us to obtain a closed form for such sequences.

To illustrate how this can be done, we consider the bouncing ball sequence:

$$2000, 1400, 980, 686, \ldots.$$

This is a geometric sequence with parameters $a = 2000$ and $r = 0.7$, which may be defined as

$$h_1 = 2000, \quad h_{n+1} = 0.7h_n \quad (n = 1, 2, 3, \ldots).$$

The way in which the terms of this sequence are obtained from the recurrence relation can be pictured as follows.

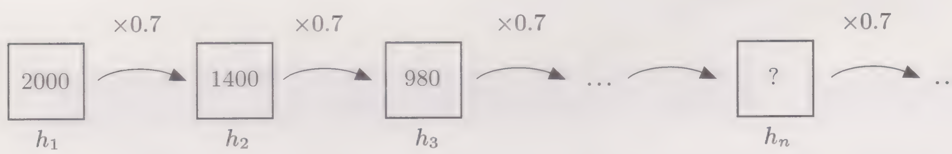


Figure 3.1 Obtaining the terms of the bouncing ball sequence

Starting from $h_1 = 2000$,

to obtain h_2, we multiply by 0.7,

to obtain h_3, we multiply by 0.7 twice,

to obtain h_4, we multiply by 0.7 three times,

and so on. To obtain the general term h_n, we have to multiply by 0.7 exactly $n - 1$ times; that is, we multiply by $(0.7)^{n-1}$. This gives the value of the general term as:

It is helpful to put fractions and decimals that are raised to a power in brackets. It is essential to do this when negative fractions and decimals are raised to a power.

$$h_n = 2000 \times (0.7)^{n-1} = 2000(0.7)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

For example, using this closed form we find that $h_4 = 2000(0.7)^3 = 686$, as expected.

Activity 3.3 Obtaining closed forms for geometric sequences

Use similar reasoning to obtain a closed form for the savings account sequence

$$s_1 = 1000, \quad s_{n+1} = 1.05s_n \quad (n = 1, 2, 3, 4).$$

Check that your closed form gives the correct value for s_4.

A solution is given on page 44.

The same reasoning can be applied to a general geometric sequence

$$x_1 = a, \quad x_{n+1} = rx_n \quad (n = 1, 2, 3, \ldots).$$

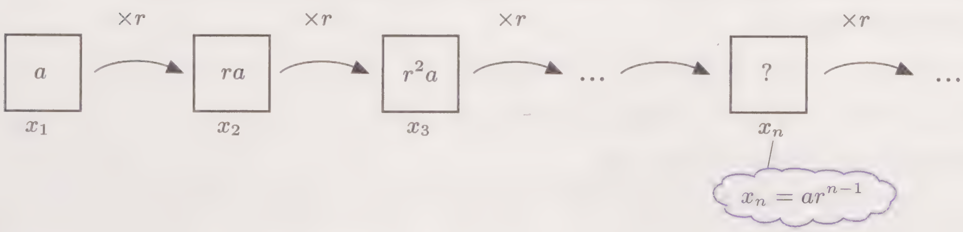


Figure 3.2 Obtaining the terms of a general geometric sequence

To obtain a general term x_n, we start with $x_1 = a$ and multiply by r exactly $n - 1$ times, so $x_n = ar^{n-1}$.

> **Closed form for geometric sequences**
>
> The geometric sequence with recurrence system
>
> $$x_1 = a, \quad x_{n+1} = rx_n \quad (n = 1, 2, 3, \ldots),$$
>
> has the closed form
>
> $$x_n = ar^{n-1} \quad (n = 1, 2, 3, \ldots).$$

Once again, we should check that this formula gives the correct answer in simple cases. For example, in the cases $n = 1$ and $n = 2$, the formula gives the correct values $x_1 = a$ and $x_2 = ar$. Also, if the ratio is $r = 1$, then we obtain the constant sequence $x_n = a$ $(n = 1, 2, 3, \ldots)$, as expected.

The next activity provides the opportunity for you to apply this closed form to particular sequences.

Activity 3.4 Writing down closed forms

Write down a closed form for each of the sequences in Activity 3.2. In each case check that your answer gives the correct value for the fourth term, and also calculate the 10th term of each sequence (correct to four significant figures).

Solutions are given on page 44.

3.3 An alternative closed form

Once again, it is not always convenient for the first term of a sequence to have subscript 1. A geometric sequence with first term x_0 is of the form

$$x_0 = a, \quad x_{n+1} = rx_n \quad (n = 0, 1, 2, \ldots).$$

To obtain x_n, we start with $x_0 = a$ and multiply by r exactly n times to give the closed form

$$x_n = ar^n \quad (n = 0, 1, 2, \ldots).$$

For example,

$$x_1 = ar,$$
$$x_2 = ar^2.$$

Activity 3.5 Using the alternative closed form

Write down a closed form for each of the following geometric sequences, and hence calculate the 10th term of each sequence (correct to four significant figures).

(a) $x_0 = 1, \quad x_{n+1} = 1.01x_n \quad (n = 0, 1, 2, \ldots)$

(b) $y_0 = 10, \quad y_{n+1} = -1.5y_n \quad (n = 0, 1, 2, \ldots)$

Solutions are given on page 44.

Summary of Section 3

This section has introduced:

◇ geometric sequences defined by recurrence systems of the form

$$x_1 = a, \quad x_{n+1} = rx_n \quad (n = 1, 2, 3, \ldots),$$

which have closed form

$$x_n = ar^{n-1} \quad (n = 1, 2, 3, \ldots);$$

◇ geometric sequences defined by recurrence systems of the form

$$x_0 = a, \quad x_{n+1} = rx_n \quad (n = 0, 1, 2, \ldots),$$

which have closed form

$$x_n = ar^n \quad (n = 0, 1, 2, \ldots);$$

◇ a method for finding the parameters a and r of a given geometric sequence.

Exercises for Section 3

Exercise 3.1

Each of the sequences in this exercise is geometric and infinite.

(a) For the sequence x_n, whose first four terms are $2, -3, 4.5, -6.75$,

(i) find the values of the parameters a and r, and write down the corresponding recurrence system;

(ii) calculate the next two terms and plot a graph of the first six terms.

(b) Repeat part (a) for the sequence y_n, whose first four terms are 100, 99, 98.01, 97.0299.

In part (a)(ii), round the two terms to three decimal places. In part (b)(ii), round the two terms to three significant figures.

Exercise 3.2

Write down a closed form for each of the following sequences, and hence calculate the 10th term of each sequence (correct to four significant figures).

(a) $x_1 = 9, \quad x_{n+1} = -\frac{2}{3}x_n \quad (n = 1, 2, 3, \ldots)$

(b) $y_0 = 1, \quad y_{n+1} = 1.02y_n \quad (n = 0, 1, 2, \ldots)$

4 Linear recurrence sequences

The sequences in this section are rather more complicated since they combine features of both arithmetic and geometric sequences. These sequences arise in correspondingly more complicated real-world contexts.

4.1 What is a linear recurrence sequence?

We begin once again with two particular sequences.

First, consider the sequence

$$6000,\ 6400,\ 6860,\ 7389,\ 7997,\ldots.$$

This represents the deer population in a colony for which it is assumed that the annual increase in population from natural causes (births minus deaths) is 15%, and this is offset by a policy of culling 500 deer every year. The figure of 15% is based on historical observations, whereas the figure of 500 has been determined as a way to control the population. We shall call this sequence P_n, so that $P_1 = 6000$, $P_2 = 6400$, $P_3 = 6860$, and so on.

To get from any term in this sequence to the next, we first *multiply* by a particular number and then *add* another:

$$6400 = 1.15 \times 6000 - 500,$$

$$6860 = 1.15 \times 6400 - 500,$$

$$7389 = 1.15 \times 6860 - 500,$$

$$7997 = 1.15 \times 7389 - 500,$$

These numbers are rounded to the nearest integer, where necessary.

and so on. The numbers 1.15 and -500 occur here because each year the natural increase is 0.15 (15%) of the population and there is a reduction of 500. Thus this sequence can be defined by the recurrence system

$$P_1 = 6000, \quad P_{n+1} = 1.15P_n - 500 \quad (n = 1, 2, 3, \ldots).$$

Next, consider the sequence

$$10\,000,\ 9697.57,\ 9380.02,\ 9046.59,\ \ldots,\ 764.09.$$

This represents the amount (in £) owing to a bank on 1 January of each year over the 20-year period of a mortgage, with fixed interest rate of 5% and total annual payment (interest plus partial repayment) of £802.43. We shall call this sequence m_n, so that $m_1 = 10\,000$, $m_2 = 9697.57$, $m_3 = 9380.02$, and so on.

Once again, to get from any term in this sequence to the next, we first *multiply* by a particular number and then *add* another:

$$9697.57 = 1.05 \times 10\,000 - 802.43,$$

$$9380.02 = 1.05 \times 9697.57 - 802.43,$$

$$9046.59 = 1.05 \times 9380.02 - 802.43,$$

These numbers are rounded to the nearest penny, where necessary.

and so on. The number 1.05 appears here because the interest to be paid each year is 0.05 times the amount owing at the beginning of the year. The apparently arbitrary number 802.43 has been carefully calculated by the bank so that at the end of the 20th year the amount owing has reduced to 0. Thus this sequence can be defined by the same kind of recurrence

system: to get from one term in the sequence to the next, we first *multiply* by 1.05 and then *add* -802.43:

$$m_1 = 10\,000, \quad m_{n+1} = 1.05m_n - 802.43 \quad (n = 1, 2, 3, \ldots, 19).$$

If you calculate a further term of this sequence, then you will find that the borrower actually overpays by 14p over the 20-year period!

The final value in the range of n is 19 because the final term in the mortgage sequence is m_{20}, the amount owing at the beginning of the 20th year.

Any sequence with the structure of the deer population and mortgage sequences is called a **linear recurrence sequence**. Thus a general linear recurrence sequence is given by:

$$x_1 = a, \quad x_{n+1} = rx_n + d \quad (n = 1, 2, 3, \ldots),$$

The name arises because the recurrence relation

$$x_{n+1} = rx_n + d$$

looks like the equation of a line:

$$y = rx + d.$$

where a is the first term. Choosing the values of a, r and d determines a particular linear recurrence sequence; we call a, r and d the **parameters** of the linear recurrence sequence. For example, the deer population sequence has $a = 6000$, $r = 1.15$ and $d = -500$, and the mortgage sequence has $a = 10\,000$, $r = 1.05$ and $d = -802.43$.

Once again, there are minor variations; for example, the range of values of the subscript n may be finite, as in the mortgage sequence, or the first term may be x_0:

$$x_0 = a, \quad x_{n+1} = rx_n + d \quad (n = 0, 1, 2, \ldots).$$

Notice that

- ◇ arithmetic sequences ($x_{n+1} = x_n + d$) are special cases of linear recurrence sequences, in which $r = 1$;
- ◇ geometric sequences ($x_{n+1} = rx_n$) are special cases of linear recurrence sequences, in which $d = 0$.

Expressed another way, linear recurrence sequences are a generalisation of both arithmetic sequences and geometric sequences.

Activity 4.1 Recognising linear recurrence sequences

Which of the following is a linear recurrence sequence? For each linear recurrence sequence, write down the values of a, r and d.

(a) $x_1 = 1, \quad x_{n+1} = 2x_n + 1 \quad (n = 1, 2, 3, \ldots)$

(b) $y_1 = 1, \quad y_{n+1} = -y_n - 1 \quad (n = 1, 2, 3, \ldots)$

(c) $z_0 = 2, \quad z_{n+1} = z_n - 1.5 \quad (n = 0, 1, 2, \ldots)$

Solutions are given on page 44.

4.2 Working with linear recurrence sequences

To study this subsection you will need a cassette player and Audio Tape 1.

In this subsection, we consider the following two questions.

- ◇ Given the first few terms of a linear recurrence sequence, is it possible to find the parameters of that sequence? For example, you may spot that

  $$1, 3, 7, 15, 31,$$

  are the first five terms of the linear recurrence sequence in Activity 4.1(a), which has parameters $a = 1$, $r = 2$ and $d = 1$. But what are the parameters of the linear recurrence sequence whose first four terms are

  $$4, -5, 13, -23?$$

- ◇ Is there a closed form for a general linear recurrence sequence? For example, you may be able to guess a closed form for

  $$1, 3, 7, 15, 31, \ldots,$$

  but what about

  $$4, -5, 13, -23, \ldots?$$

Listen to Audio Tape 1, Band 1, 'Working with linear recurrence sequences', in which both these questions are answered. This tape refers to the following frames, Frame 2 of which contains Activity 4.2.

Frame 1

Finding parameters for linear recurrence sequences

Find the linear recurrence sequence which begins:

4, –5, 13, –23.

General linear recurrence sequence:

$$x_1 = a, \quad x_{n+1} = rx_n + d \quad (n = 1, 2, 3, \ldots). \tag{1}$$

In this case $x_1 = 4$, $x_2 = -5$, $x_3 = 13$, $x_4 = -23$. So

$a = 4$

Now apply equation (1) with $n = 1, 2$:

$$\begin{cases} x_2 = rx_1 + d \\ x_3 = rx_2 + d \end{cases} \quad \text{gives} \quad \begin{cases} -5 = 4r + d & (2) \\ 13 = -5r + d & (3) \end{cases}$$

Subtract equation (3) from equation (2):

$$-5 - 13 = (4r + d) - (-5r + d);$$

that is,

$$-18 = 9r,$$

so

$r = -2.$

Simultaneous equations — solved by elimination

Find d from equation (2): $d = -5 - 4r = -5 - 4(-2)$, so

$d = 3.$

Hence the linear recurrence sequence is:

$$x_1 = 4, \quad x_{n+1} = -2x_n + 3 \quad (n = 1, 2, 3, \ldots).$$

Check

This gives

$$x_4 = -2x_3 + 3 = -2 \times 13 + 3 = -23, \text{ as expected.}$$

Frame 2

Activity 4.2 Finding parameters

Find the linear recurrence sequences that begin:

(a) 3, 5, 9, 17; (b) 12, –3, 2, 1/3; (c) 0, 2, 0, 2.

Also, for each sequence, check the fourth term, calculate two further terms and plot its graph.

Solutions are on page 44.

Frame 3

Closed form — a special case

Specialising: $a = 1, d = 1$.

Is there a closed form for

$$x_1 = 1, \quad x_{n+1} = rx_n + 1 \quad (n = 1, 2, 3, \ldots)?$$

$r \neq 1$, or else arithmetic

First few terms:

$$x_1 = 1,$$
$$x_2 = rx_1 + 1 = r + 1,$$
$$x_3 = rx_2 + 1 = r(r+1) + 1 = r^2 + r + 1,$$
$$x_4 = rx_3 + 1 = r(r^2 + r + 1) + 1 = r^3 + r^2 + r + 1,$$
$$\vdots$$
$$x_{10} = r^9 + r^8 + \cdots + r + 1.$$

Pattern continues

In general,

$$x_n = r^{n-1} + r^{n-2} + \cdots + r + 1. \quad (1)$$

To obtain a neat form for x_n:

$$rx_n = r^n + \boxed{r^{n-1} + \cdots + r^2 + r}. \quad (2)$$

Subtract equation (1) from equation (2) to cancel most terms:

$$rx_n - x_n = r^n - 1.$$

So

$$\boxed{x_n = \frac{r^n - 1}{r - 1} \quad (n = 1, 2, 3, \ldots). \quad r \neq 1} \quad (3)$$

Does this closed form generalise?

Remark Combining equations (1) and (3) gives

$$r^{n-1} + r^{n-2} + \cdots + r + 1 = \frac{r^n - 1}{r - 1}, \quad r \neq 1. \quad (4)$$

This formula will be used in Frames 4 and 5.

Frame 4

Closed form — the general case

Generalising

Is there a closed form for

$$x_1 = a, \quad x_{n+1} = rx_n + d \quad (n = 1, 2, 3, \ldots)?$$

$r \neq 1$, or else arithmetic

First few terms:

$$x_1 = a,$$
$$x_2 = ra + d,$$
$$x_3 = r(ra + d) + d = r^2a + rd + d,$$
$$x_4 = r(r^2a + rd + d) + d = r^3a + r^2d + rd + d,$$
$$\vdots$$
$$x_n = r^{n-1}a + (r^{n-2} + \cdots + r + 1)d,$$
$$= r^{n-1}a + \left(\frac{r^{n-1}-1}{r-1}\right)d.$$

Frame 3, equation (4) with n replaced by $n-1$.

So

$$\boxed{x_n = \left(a + \frac{d}{r-1}\right)r^{n-1} - \frac{d}{r-1} \quad (n = 1, 2, 3, \ldots) \quad r \neq 1} \qquad (1)$$

Example

The linear recurrence sequence

$$x_1 = 4, \quad x_{n+1} = -2x_n + 3 \quad (n = 1, 2, 3, \ldots)$$

Frame 1

has $a = 4$, $r = -2$ and $d = 3$. Equation (1) gives

$$x_n = \left(4 + \frac{3}{-2-1}\right)(-2)^{n-1} - \frac{3}{-2-1}$$
$$= 3(-2)^{n-1} + 1 \quad (n = 1, 2, 3, \ldots).$$

Check that this closed form does give $x_1 = 4$, $x_2 = -5$, $x_3 = 13$, $x_4 = -23$.

Frame 5

Sum of a finite geometric series

$$S = a + ar + ar^2 + \cdots + ar^n$$

$r \neq 1$

$$= a(1 + r + r^2 + \cdots + r^n).$$

From Frame 3, equation (4), with n replaced by $n+1$

$$\boxed{S = a\left(\frac{r^{n+1}-1}{r-1}\right) \quad \text{or} \quad S = a\left(\frac{1-r^{n+1}}{1-r}\right).}$$

Activity 4.3 Writing down closed forms

Write down a closed form for each of the sequences in Activity 4.2, and hence calculate the 10th term of each sequence.

Solutions are given on page 45.

Activity 4.4 Checking simple cases

Check that the closed form for x_n obtained in equation (1) of Frame 4 gives the correct values in the special cases $n = 1$ and $n = 2$.

Solutions are given on page 46.

4.3 An alternative closed form

Once again, it is not always convenient for the first term of a sequence to have subscript 1. If we consider a linear recurrence sequence of the form

$$x_0 = a, \quad x_{n+1} = rx_n + d \quad (n = 0, 1, 2, \ldots),$$

where $r \neq 1$, then the closed form for x_n is

If $r = 1$, then the sequence is arithmetic.

$$x_n = \left(a + \frac{d}{r-1}\right) r^n - \frac{d}{r-1} \quad (n = 0, 1, 2, \ldots).$$

As before this version of the closed form is slightly neater than the one with first term x_1.

Activity 4.5 Using the alternative closed form

Write down a closed form for each of the following linear recurrence sequences, and hence calculate the 10th term of each sequence (correct to three decimal places, where appropriate).

(a) $x_0 = 50, \quad x_{n+1} = 0.9x_n + 8 \quad (n = 0, 1, 2, \ldots)$

(b) $y_0 = -1, \quad y_{n+1} = -2y_n + 1 \quad (n = 0, 1, 2, \ldots)$

(c) $z_0 = 1, \quad z_{n+1} = z_n + \frac{1}{2} \quad (n = 0, 1, 2, \ldots)$

Solutions are given on page 46.

Summary of Section 4

This section has introduced:

- linear recurrence sequences defined by recurrence systems of the form

  $$x_1 = a, \quad x_{n+1} = rx_n + d \quad (n = 1, 2, 3, \ldots),$$

  which, when $r \neq 1$, have closed form

  $$x_n = \left(a + \frac{d}{r-1}\right) r^{n-1} - \frac{d}{r-1} \quad (n = 1, 2, 3, \ldots);$$

- linear recurrence sequences defined by recurrence systems of the form

  $$x_0 = a, \quad x_{n+1} = rx_n + d \quad (n = 0, 1, 2, \ldots),$$

  which, when $r \neq 1$, have closed form

  $$x_n = \left(a + \frac{d}{r-1}\right) r^{n} - \frac{d}{r-1} \quad (n = 0, 1, 2, \ldots);$$

- a method for finding the parameters of a linear recurrence sequence whose first three terms are known;
- the formula for the sum of the finite geometric series $a + ar + ar^2 + \cdots + ar^n$, which is

  $$a + ar + ar^2 + \cdots + ar^n = a\left(\frac{1 - r^{n+1}}{1 - r}\right), \ r \neq 1.$$

Exercises for Section 4

Exercise 4.1

Each of the sequences in this exercise is an infinite linear recurrence sequence.

(a) For the sequence x_n, whose first four terms are 1, 3.1, 5.41, 7.951,

(i) find the values of the parameters a, r and d, and write down the corresponding recurrence system;

(ii) calculate the next two terms and plot a graph of the first six terms.

(b) Repeat part (a) for the sequence y_n, whose first four terms are 4, 3, 2.75, 2.6875.

Exercise 4.2

Write down a closed form for each of the following sequences, and hence calculate the 10th term of each sequence.

(a) $x_1 = 0, \quad x_{n+1} = 3x_n + 1 \quad (n = 1, 2, 3, \ldots)$

(b) $y_0 = 1, \quad y_{n+1} = \frac{1}{2}y_n + 1 \quad (n = 0, 1, 2, \ldots)$

(c) $z_0 = 100, \quad z_{n+1} = z_n - 0.1 \quad (n = 0, 1, 2, \ldots)$

Exercise 4.3

Write down a closed form for

(a) the deer population sequence P_n,

(b) the mortgage sequence m_n.

These sequences are given in Subsection 4.1.

5 Long-term behaviour of sequences

In this section we study the long-term behaviour of linear recurrence sequences.

5.1 Types of long-term behaviour

In Activity 4.5, you found that the linear recurrence sequences

$$x_0 = 50, \quad x_{n+1} = 0.9x_n + 8 \quad (n = 0, 1, 2, \ldots),$$

$$y_0 = -1, \quad y_{n+1} = -2y_n + 1 \quad (n = 0, 1, 2, \ldots),$$

$$z_0 = 1, \quad z_{n+1} = z_n + \tfrac{1}{2} \quad (n = 0, 1, 2, \ldots),$$

have closed forms

$$x_n = -30(0.9)^n + 80 \quad (n = 0, 1, 2, \ldots),$$

$$y_n = -\tfrac{4}{3}(-2)^n + \tfrac{1}{3} \quad (n = 0, 1, 2, \ldots),$$

$$z_n = 1 + \frac{n}{2} \quad (n = 0, 1, 2, \ldots),$$

respectively. For these sequences, we can use either the recurrence relation or the closed form to calculate the following tables of values and hence obtain the corresponding graphs.

Table 5.1 The sequence x_n (to 1 d.p.)

n	0	1	2	3	4	5
x_n	50	53	55.7	58.1	60.3	62.3

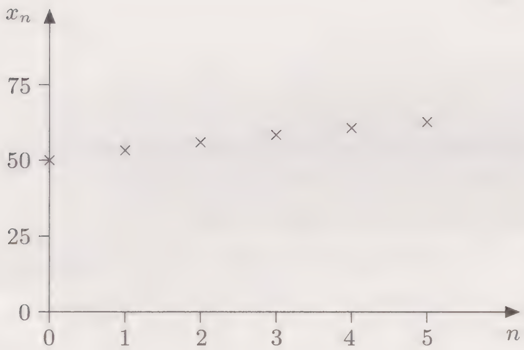


Figure 5.1 The sequence x_n

Table 5.2 The sequence y_n

n	0	1	2	3	4	5
y_n	-1	3	-5	11	-21	43

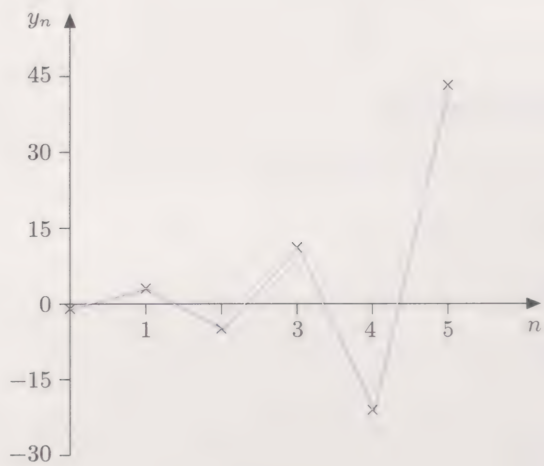


Figure 5.2 The sequence y_n

Table 5.3 The sequence z_n

n	0	1	2	3	4	5
z_n	1	1.5	2	2.5	3	3.5

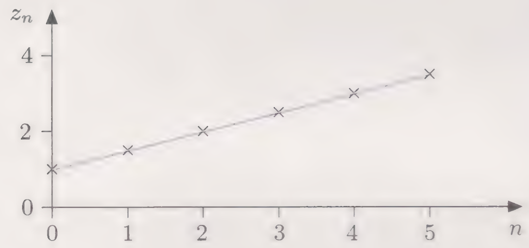


Figure 5.3 The sequence z_n

Seeing the first few terms of these sequences plotted on a graph gives you some idea of their **long-term behaviour**; that is, how each of these sequences will develop as more and more terms are considered.

◇ The early terms of the sequence x_n are increasing steadily and it seems likely that its terms will continue to increase, but will they get very large or level out?

◇ The early terms of the sequence y_n oscillate between positive and negative values, getting larger and larger, but will this behaviour continue?

◇ All the terms of the sequence z_n lie on a straight line, rising from left to right, and so as n gets large so does the value of the term z_n.

It is possible to infer the behaviour of all the terms of z_n because the closed form $z_n = 1 + n/2$ can be associated with the equation of a straight line. The closed forms for the sequences x_n and y_n depend on $(0.9)^n$ and $(-2)^n$, and are not so easy to interpret. These matters are addressed in the next subsection.

5.2 *The long-term behaviour of r^n*

In general, in the closed form for a linear recurrence sequence, the only part of the formula for the general term x_n that changes when n changes is the expression r^{n-1} (or r^n if the first term is x_0). Thus the long-term behaviour of the sequence x_n depends on the long-term behaviour of the sequence r^n. To gain some insight into the long-term behaviour of r^n $(n = 0, 1, 2, \ldots)$ consider the following table of values of r^n, for $r = 0.5, 0.9, 1, 1.1$ and 1.5.

See Subsections 4.2 and 4.3.

In the special case $r = 1$, we have $r^n = 1$ and the long-term behaviour of this constant sequence is easy to describe. Here we consider some (positive) values for r, on either side of 1.

Table 5.4 Values of r^n

n	0	1	2	3	4	5	$\ldots$	100
$(0.5)^n$	1	0.5	0.25	0.125	0.0625	0.031 25	$\ldots$	7.89×10^{-31}
$(0.9)^n$	1	0.9	0.81	0.729	0.6561	0.590 49	$\ldots$	2.66×10^{-5}
1^n	1	1	1	1	1	1	$\ldots$	1
$(1.1)^n$	1	1.1	1.21	1.331	1.4641	1.610 51	$\ldots$	1.38×10^4
$(1.5)^n$	1	1.5	2.25	3.375	5.0625	7.593 75	$\ldots$	4.07×10^{17}

The entries in the final column of the table are given to three significant figures.

In each of the first two rows of this table, the values decrease from left to right. This happens because if r is between 0 and 1, but not equal to these values, then

$r = 0.5, 0.9$

$$r^{n+1} = r \times r^n < r^n.$$

On the other hand, in the last two rows the values increase from left to right because in each of these cases $r > 1$.

$r = 1.1, 1.5$

Moreover, on the basis of the entries in the final column, it seems plausible that if $0 < r < 1$, then the terms r^n become *arbitrarily small* as n becomes large; that is, r^n may be made as small as we please by taking n large enough. On the other hand, if $r > 1$, then the terms r^n become *arbitrarily large* as n becomes large. We shall treat these as 'known properties' of the sequence r^n and use the following standard notation to describe them:

These properties are established in courses on real analysis.

$$\text{if } 0 < r < 1, \text{ then } r^n \to 0 \text{ as } n \to \infty;$$
$$\text{if } r > 1, \text{ then } r^n \to \infty \text{ as } n \to \infty.$$

In each case the arrow is read as 'tends to' and the symbol ∞ as 'infinity'.

Using this, we can deduce the long-term behaviour of

$$x_n = -30(0.9)^n + 80 \quad (n = 0, 1, 2, \ldots).$$

As n becomes large, $(0.9)^n$ becomes arbitrarily small (because $0 < 0.9 < 1$), so that $30(0.9)^n$ also becomes arbitrarily small, and hence x_n becomes arbitrarily close to 80. This answers the question about the sequence x_n posed after Figure 5.3. The long-term behaviour of the sequence x_n is that the terms rise towards 80 but get no higher, so they must level out. We can express this long-term behaviour as

In symbols, $(0.9)^n \to 0$ as $n \to \infty$.

$$x_n \to 80 \text{ as } n \to \infty.$$

We now use the fact that if $r > 1$, then the values of r^n become arbitrarily large as n becomes large, to deduce the long-term behaviour of

$$y_n = -\tfrac{4}{3} \times (-2)^n + \tfrac{1}{3} \quad (n = 0, 1, 2, \ldots).$$

Since $-2 = (-1) \times 2$, we have

$$(-2)^n = (-1)^n \times 2^n.$$

First, we write $(-2)^n$ in the form $(-2)^n = (-1)^n \times 2^n$. As n becomes large, 2^n becomes arbitrarily large, because $2 > 1$. In symbols, $2^n \to \infty$ as $n \to \infty$. Thus $(-2)^n$ becomes arbitrarily large, but alternates in sign (switching between positive and negative). Multiplying $(-2)^n$ by $-\frac{4}{3}$ and adding $\frac{1}{3}$ to the result does not change this overall behaviour of $(-2)^n$ as n gets large, so y_n also has these properties, as suggested after Figure 5.3. Any sequence whose terms become arbitrarily large (whether positive or negative) is called **unbounded**.

The following table gives a summary of the long-term behaviour of r^n for various ranges of values of r, including negative ones.

Table 5.5 Long-term behaviour of r^n

Range of r	Behaviour of r^n
$r > 1$	$r^n \to \infty$ as $n \to \infty$
$r = 1$	Remains constant: 1, 1, 1, ...
$0 < r < 1$	$r^n \to 0$ as $n \to \infty$
$r = 0$	Remains constant: 0, 0, 0, ...
$-1 < r < 0$	$r^n \to 0$ as $n \to \infty$, alternates in sign
$r = -1$	Alternates between -1 and $+1$
$r < -1$	r^n is unbounded, alternates in sign

Note that, for a given value of $r > 0$, the long-term behaviour of r^{n-1} $(n = 1, 2, 3, \dots)$ is the same as the long-term behaviour of r^n $(n = 0, 1, 2, \dots)$ because these two sequences have the same terms.

Activity 5.1 Long-term behaviour

Describe the long-term behaviour of each of the sequences in Activity 4.2, whose closed forms you found in Activity 4.3.

Solutions are given on page 46.

Summary of Section 5

This section has introduced various types of long-term behaviour of linear recurrence sequences.

Exercises for Section 5

Exercise 5.1

Describe the long-term behaviour of the three sequences in Exercise 4.2.

Exercise 5.2

Describe the long-term behaviour of the deer population sequence, whose closed form was found in Exercise 4.3.

6 Investigating sequences with the computer

In this section you will need computer access, the files for this chapter and Computer Book A.

The computer can be used to generate tables and graphs containing many terms of a sequence. This makes it possible to study in greater detail than in Section 5 the long-term behaviour of sequences.

Refer to Computer Book A for the work in this section.

Summary of Section 6

This section has used the computer to confirm the long-term behaviour of various linear recurrence sequences.

7 Sequences and modelling

In this chapter, we have studied several sequences which are defined by recurrence systems and which arise as descriptions of what occurs in the real world.

- The book sequence

$$b_1 = 5, \quad b_{n+1} = b_n + 3 \quad (n = 1, 2, 3, \ldots, 11).$$

- The oil sequence

$$v_1 = 1000, \quad v_{n+1} = v_n - 50 \quad (n = 1, 2, 3, \ldots, 20).$$

- The savings account sequence

$$s_1 = 1000, \quad s_{n+1} = 1.05s_n \quad (n = 1, 2, 3, 4).$$

- The bouncing ball sequence

$$h_1 = 2000, \quad h_{n+1} = 0.7h_n \quad (n = 1, 2, 3, \ldots).$$

- The deer population sequence

$$P_1 = 6000, \quad P_{n+1} = 1.15P_n - 500 \quad (n = 1, 2, 3, \ldots).$$

- The mortgage sequence

$$m_1 = 10\,000, \quad m_{n+1} = 1.05m_n - 802.43 \quad (n = 1, 2, 3, \ldots, 19).$$

In this final section, we briefly discuss the application of mathematics to the real world, and introduce one framework for doing this.

You may already have realised that the six sequences above are of two rather different types. The first type are those sequences in which the recurrence system determines *exactly* what happens in the real world. For example, the savings account sequence represents the rule designed by the bank to handle that account, and the terms of that sequence will be exact as long as the bank wishes to maintain that rule with the same interest rate.

On the other hand, it is very unlikely that our deer population sequence is an exact representation of any deer population in the real world! A few moments' thought suggests many ways in which it might prove inadequate:

- the number of deer is unlikely to be known exactly;
- the birth and death rates, which must have been calculated before culling began, are probably imprecise and may vary over time considerably;
- the number of deer may change for other reasons, such as overcrowding or migration;
- the predicted values of P_n, for $n > 4$, are not integers, so they have to be rounded.

In spite of these factors, the use of recurrence systems to attempt to predict population changes (not just in deer, of course) is of major practical importance. Any collection of formulas which attempts to quantify how some aspect of the real world behaves is called a **mathematical model** or, more briefly, a **model**. So our deer population sequence P_n is a simple model. It does not represent the real world exactly, but it may represent it sufficiently accurately for some practical purpose, and if so it has the advantage of being simple.

In addition to predicting how some aspect of the real world *will* change, models are also used to fill in gaps in existing incomplete sets of data.

Activity 7.1 Models

Which of the six recurrence sequences at the beginning of this section are mathematical models, in the sense just described?

Solutions are given on page 46.

In this course, you will meet mathematical models often, and we shall try to use models in a practical way. This means that we aim to keep in mind the real-world purpose for which the model was created, rather than devoting our efforts exclusively to mathematical aspects of the models, fascinating though these may be. This may lead us to change a model if it proves unsatisfactory. To see how this can happen, consider the deer population model P_n:

$$P_1 = 6000, \quad P_{n+1} = 1.15P_n - 500 \quad (n = 1, 2, 3, \ldots).$$

Remember that 6000, 1.15 and -500 are the *parameters* of this linear recurrence sequence.

This model was created in order to provide a basis for controlling a certain deer population. The number 1.15 arises from the 15% natural growth rate of the population, which would have been obtained by collecting data from previous years. The number of deer to be culled per year would have been decided by calculating how the terms of the sequence P_n behave in various cases and then deciding that by culling 500 the population would not grow too quickly. However, after a few years it may turn out that the growth in the deer population is higher, or lower, than that predicted by the model. We may then wish to vary the culling figure, or perhaps choose a different type of recurrence relation, which we hope will represent the behaviour of the population more accurately. This process of choosing a model, trying it out, evaluating it, and possibly changing it, is known as the **modelling cycle**.

The framework that we use for this modelling cycle is set out below in five key stages. As you read these, you may like to picture how these general statements would apply in practice if you were given the task of modelling a real deer population.

1. **Specify the purpose:**
   define the problem; decide which aspects of the problem to investigate; collect relevant data.
2. **Create the model:**
   choose variables; state assumptions; formulate mathematical relationships.
3. **Do the mathematics:**
   solve equations; draw graphs; derive results.
4. **Interpret the results:**
   describe the mathematical solution in words; decide what results to compare with reality.
5. **Evaluate the outcomes:**
   test the outcomes of the model with reality; if necessary, adjust the model and enter the cycle again.

To help you recall these stages, the following diagram of the modelling cycle, which summarises the above list, is used consistently throughout the course.

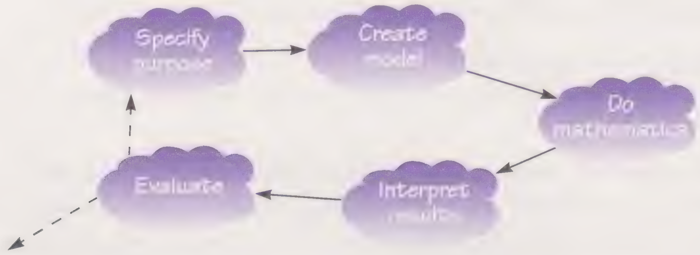


Figure 7.1 The modelling cycle

The following activity gives you a chance to get used to the *process* of modelling.

Activity 7.2 The modelling process

Write one or two sentences on how *you* think each stage of the modelling cycle might apply to the problem of modelling a bouncing ball.

Possible solutions are given on page 46.

In this course, you will meet numerous mathematical models, of varying complexity. None, however, will be remotely comparable to the highly complex mathematical models used by:

- ◇ the Treasury, to make forecasts about the behaviour of the British economy;
- ◇ the Metereological Office, to make weather forecasts.

Both these models involve hundreds of interdependent variables forming vast recurrence systems (with no simple closed form, unfortunately), coded into computer programs. Nevertheless, the underlying process used to develop such complex models must follow some modelling cycle.

Summary of Section 7

This section has introduced the framework within which the course will apply mathematics to try to solve real-world problems.

Summary of Chapter A1

In this chapter, you met several general types of sequences which can be used to model real-world processes. These sequences were defined by first-order recurrence systems, and in each case a closed form was found. These closed forms made it possible to study the long-term behaviour of the sequences.

Notice that your understanding of the various types of sequences was aided by first considering special cases. For example, we found closed forms for *particular* arithmetic, geometric and linear recurrence sequences before doing this for the *general* cases. This process of specialising and then generalising will be a recurring theme in the course.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Sequence, term, subscript notation, closed form, recurrence relation, recurrence system, recurrence sequence, arithmetic sequence, constant sequence, geometric sequence, linear recurrence sequence, parameters of a linear recurrence sequence, finite geometric series, long-term behaviour of a sequence.

Symbols and notation to know and use

◇ a_n for a sequence, and for its term with subscript n.

◇ $a_n = n^2 \quad (n = 1, 2, 3, \ldots)$ for a closed form.

◇ $a_1 = 0, \quad a_{n+1} = 2a_n - 1 \quad (n = 1, 2, 3, \ldots)$, for a recurrence system.

◇ $a_n \to l$ as $n \to \infty$; $a_n \to \infty$ as $n \to \infty$.

Mathematical skills

◇ Consider special cases to get a sense of what may happen in general.

◇ Obtain a closed form for a linear recurrence sequence by manipulating algebraic expressions arising from known terms.

◇ Apply closed forms for linear recurrence sequences in particular cases.

◇ Describe the long-term behaviour of a given linear recurrence sequence.

Modelling skills

◇ Be aware that recurrence systems can be used as models to try to quantify changes in the real-world.

◇ Be aware that a systematic framework is used within which to perform modelling.

Mathcad skills

- ◇ Create a sequence and display tables of values.
- ◇ Plot sequences from closed forms.
- ◇ Use tables and plots to confirm the long-term behaviour of sequences.

Ideas to be aware of

- ◇ Different real-world contexts may lead to similar mathematical models.
- ◇ The behaviour of linear recurrence sequences varies according to the values of the parameters.
- ◇ Some recurrence systems are used to *design* aspects of the real world, whereas others are used to *model* aspects of the real world.

Solutions to Activities

Solution 1.1

(a) $t_2 = 12$

(b) $t_6 = 9$

Solution 1.2

(a) $a_1 = 7,\ a_2 = 14,\ a_3 = 21,\ a_4 = 28,\ a_5 = 35,$
$a_{100} = 700.$

(b) $b_1 = 1,\ b_2 = 1/2,\ b_3 = 1/3,\ b_4 = 1/4,\ b_5 = 1/5,$
$b_{100} = 1/100.$

(You may have converted these simple fractions to decimals, but there is no need to do this in such cases. If you did the conversions, you should have used the default precision of three decimal places.)

(c) $c_1 = (-1)^{1+1} = (-1)^2 = 1,$
$c_2 = (-1)^{2+1} = (-1)^3 = -1,$
$c_3 = 1,\ c_4 = -1,\ c_5 = 1,\ c_{100} = -1.$

Solution 1.3

(a) Since $1, 8, 27, 64$ are all cubes of natural numbers:

$$1 = 1^3, 8 = 2^3, 27 = 3^3, 64 = 4^3,$$

a suitable closed form is

$$a_n = n^3 \quad (n = 1, 2, 3, \ldots).$$

(b) Hence $a_{10} = 10^3 = 1000.$

(Another closed form that gives the four terms $1, 8, 27, 64$ is

$$b_n = n^3 + (n-1)(n-2)(n-3)(n-4).$$

In this case the tenth term is

$$b_{10} = 4024.$$

The two closed forms specify different sequences.)

Solution 1.4

(a) $a_0 = 3^0 = 1,\ a_1 = 3^1 = 3,\ a_2 = 3^2 = 9.$

(b) $b_2 = 1/2,\ b_3 = 1/6,\ b_4 = 1/12.$

(c) $c_1 = 1/2,\ c_2 = 1/6,\ c_3 = 1/12.$

Solution 1.5

(a)

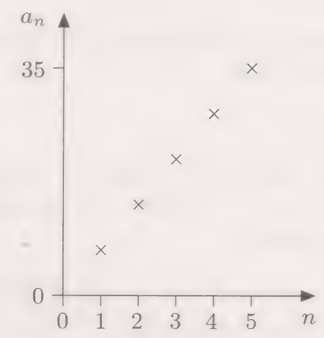


Figure S.1

(b)

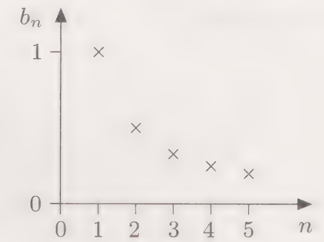


Figure S.2

(c) The lines joining the points in this graph draw attention to the fact that the terms in this sequence alternate in sign.

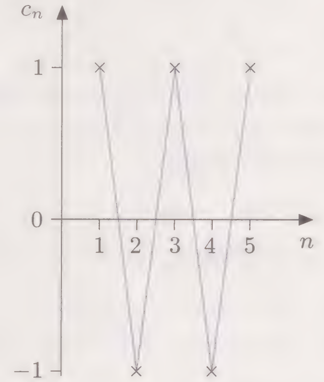


Figure S.3

Solution 1.6

(a) $a_1 = 0,\ a_2 = 1,\ a_3 = 3,\ a_4 = 7,\ a_5 = 15.$

(b) $b_1 = 1,\ b_2 = 0,\ b_3 = -1,\ b_4 = 0,\ b_5 = -1.$

(c) $c_0 = 2,\ c_1 = 1.5,\ c_2 = 1.41\dot{6},$
$c_3 = 1.414\,216$ (to 6 d.p.),
$c_4 = 1.414\,214$ (to 6 d.p.).

Solution 2.1

(a) The sequence x_n is arithmetic, with parameters $a = -1$ and $d = 1$.

(b) The sequence y_n is not arithmetic. The term y_n on the right of the recurrence relation is multiplied by -1.

(c) The sequence z_n is arithmetic, with parameters $a = 1$ and $d = -1.5$.

Solution 2.2

(a) (i) The first term is $a = 1$ and the difference is $d = 3$ $(= 4 - 1 = 7 - 4 = 10 - 7)$. So the recurrence system is

$$x_1 = 1, \quad x_{n+1} = x_n + 3 \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$x_5 = x_4 + 3 = 10 + 3 = 13,$$
$$x_6 = x_5 + 3 = 13 + 3 = 16.$$

The graph is as follows.

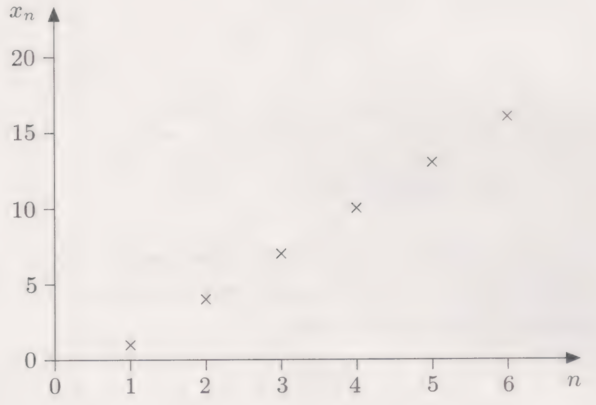


Figure S.4

(b) (i) The first term is $a = 2.1$ and the difference is $d = 1.1$ $(= 3.2 - 2.1 = 4.3 - 3.2 = 5.4 - 4.3)$. So the recurrence system is

$$y_1 = 2.1, \quad y_{n+1} = y_n + 1.1 \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$y_5 = y_4 + 1.1 = 5.4 + 1.1 = 6.5,$$
$$y_6 = y_5 + 1.1 = 6.5 + 1.1 = 7.6.$$

The graph is as follows.

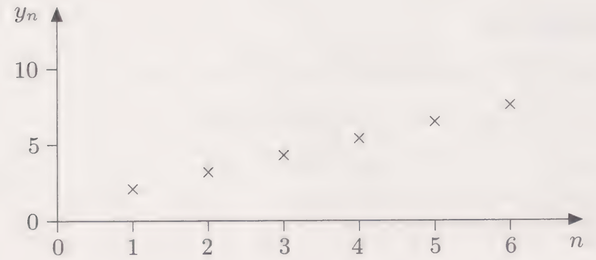


Figure S.5

(c) (i) The first term is $a = 1$ and the difference is $d = -0.1$ $(= 0.9 - 1 = 0.8 - 0.9 = 0.7 - 0.8)$. So the recurrence system is

$$z_1 = 1, \quad z_{n+1} = z_n - 0.1 \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$z_5 = z_4 - 0.1 = 0.7 - 0.1 = 0.6,$$
$$z_6 = z_5 - 0.1 = 0.6 - 0.1 = 0.5.$$

The graph is as follows.

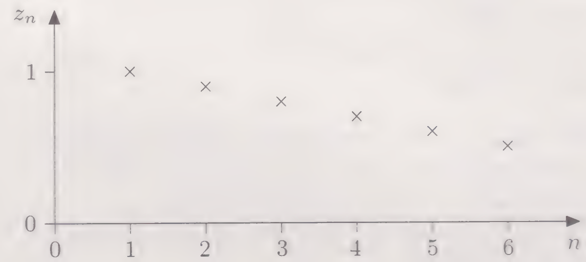


Figure S.6

In each of the graphs, the points plotted lie on a straight line. The first two graphs are increasing, from left to right, and the last one is decreasing. This happens because, in each case, from one point to the next the horizontal change is always 1 and the vertical change is always d, the difference.

Solution 2.3

Starting from $v_1 = 1000$,

to obtain v_2, we subtract 50,

to obtain v_3, we subtract 50 twice,

to obtain v_4, we subtract 50 three times,

and so on. Thus, to obtain the general term x_n, we have to subtract 50 exactly $n - 1$ times; that is, we subtract $50(n - 1)$. This gives the value of the general term as:

$$\begin{aligned} v_n &= 1000 - 50(n - 1) \\ &= 1050 - 50n \quad (n = 1, 2, 3, \ldots, 21). \end{aligned}$$

This gives $v_4 = 1050 - 50 \times 4 = 850$, as expected.

Solution 2.4

(a) Since $a = 1$ and $d = 3$, the closed form is

$$x_n = 1 + 3(n - 1) = 3n - 2 \quad (n = 1, 2, 3, \ldots).$$

This gives $x_4 = 3 \times 4 - 2 = 10$, as expected, and also

$$x_{100} = 3 \times 100 - 2 = 298.$$

(b) Since $a = 2.1$ and $d = 1.1$, the closed form is

$$\begin{aligned} y_n &= 2.1 + 1.1(n - 1) \\ &= 1.1n + 1 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

This gives $y_4 = 1.1 \times 4 + 1 = 5.4$, as expected, and also

$$y_{100} = 1.1 \times 100 + 1 = 111.$$

(c) Since $a = 1$ and $d = -0.1$, the closed form is

$$\begin{aligned} z_n &= 1 - 0.1(n-1) \\ &= 1.1 - 0.1n \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

This gives $z_4 = 1.1 - 0.1 \times 4 = 0.7$, as expected, and also

$$z_{100} = 1.1 - 0.1 \times 100 = -8.9.$$

Solution 2.5

(a) Since $a = 1$ and $d = 2$, the alternative closed form is

$$x_n = 1 + 2n \quad (n = 0, 1, 2, \ldots).$$

Since the first term is x_0, the 100th term is

$$x_{99} = 1 + 2 \times 99 = 199.$$

(b) Since $a = 10$ and $d = -0.01$, the alternative closed form is

$$y_n = 10 - 0.01n \quad (n = 0, 1, 2, \ldots).$$

The 100th term is

$$y_{99} = 10 - 0.01 \times 99 = 9.01.$$

Solution 3.1

(a) The sequence x_n is geometric, with parameters $a = -1$ and $r = 3$.

(b) The sequence y_n is geometric, with parameters $a = 1$ and $r = -0.9$.

(c) The sequence z_n is not geometric, because the expression on the right contains the term $+1$.

Solution 3.2

(a) (i) The first term is $a = 1$ and the ratio is $r = \frac{1}{2}$ $(= \frac{1}{2}/1 = \frac{1}{4}/\frac{1}{2} = \frac{1}{8}/\frac{1}{4})$. So the recurrence system is

$$x_1 = 1, \quad x_{n+1} = \tfrac{1}{2}x_n \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$x_5 = \tfrac{1}{2}x_4 = \tfrac{1}{16},$$
$$x_6 = \tfrac{1}{2}x_5 = \tfrac{1}{32}.$$

The graph is as follows.

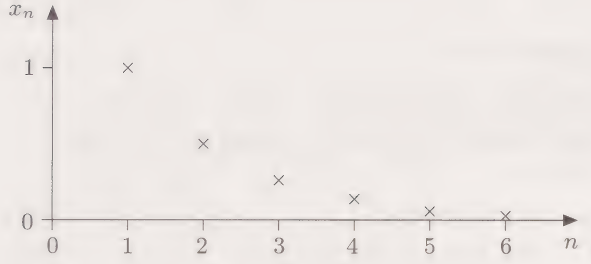


Figure S.7

(b) (i) The first term is $a = 4.2$ and the ratio is $r = 1.7$ $(= 7.14/4.2 = 12.138/7.14 = 20.6346/12.138)$. So the recurrence system is

$$y_1 = 4.2, \quad y_{n+1} = 1.7y_n \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$\begin{aligned} y_5 = 1.7y_4 &= 1.7 \times 20.6346 \\ &= 35.078\,82 \\ &= 35.079 \text{ (to 3 d.p.)}, \\ y_6 = 1.7y_5 &= 1.7 \times 35.078\,82 \\ &= 59.633\,994 \\ &= 59.634 \text{ (to 3 d.p.)}. \end{aligned}$$

The graph is as follows.

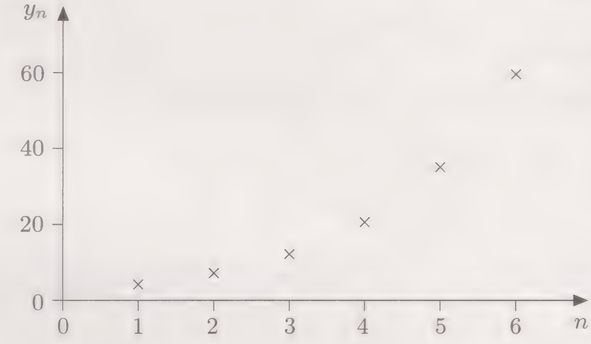


Figure S.8

(c) (i) The first term is $a = 2$ and the ratio is $r = -1$ $(= (-2)/2 = 2/(-2) = (-2)/2)$. So the recurrence system is

$$z_1 = 2, \quad z_{n+1} = -z_n \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$z_5 = -z_4 = -(-2) = 2,$$
$$z_6 = -z_5 = -2.$$

The graph is as follows.

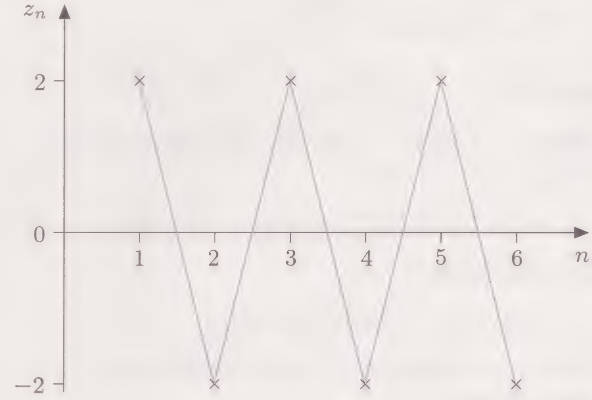


Figure S.9

The three graphs have no obvious common features.

Solution 3.3

Starting from $s_1 = 1000$,

to obtain s_2, we multiply by 1.05,

to obtain s_3, we multiply by 1.05 twice,

to obtain s_4, we multiply by 1.05 three times,

and so on. To obtain a general term s_n, we have to multiply by 1.05 exactly $n-1$ times; that is, we multiply by $(1.05)^{n-1}$. This gives the value of the general term as:

$$s_n = 1000(1.05)^{n-1} \quad (n = 1, 2, 3, 4, 5).$$

Using this closed form we find that $s_4 = 1000(1.05)^3 = 1157.63$ to the nearest penny, as expected.

Solution 3.4

(a) Since $a = 1$ and $r = \frac{1}{2}$, the closed form is

$$\begin{aligned} x_n &= 1 \times \left(\tfrac{1}{2}\right)^{n-1} \\ &= \left(\tfrac{1}{2}\right)^{n-1} \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

This gives $x_4 = (\frac{1}{2})^3 = \frac{1}{8}$, as expected, and also

$$\begin{aligned} x_{10} = \left(\tfrac{1}{2}\right)^9 &= \frac{1}{512} \\ &= 1.953 \times 10^{-3} \text{ (to 4 s.f.)}. \end{aligned}$$

(b) Since $a = 4.2$ and $r = 1.7$, the closed form is

$$y_n = 4.2(1.7)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

This gives $y_4 = 4.2 \times 1.7^3 = 20.6346$, as expected, and also

$$y_{10} = 4.2 \times 1.7^9 = 498.1 \text{ (to 4 s.f.)}.$$

(c) Since $a = 2$ and $r = -1$, the closed form is

$$z_n = 2 \times (-1)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

This gives $z_4 = 2 \times (-1)^3 = -2$, as expected, and also

$$z_{10} = 2 \times (-1)^9 = -2.$$

Solution 3.5

(a) Since $a = 1$ and $r = 1.01$, the alternative closed form is

$$x_n = 1 \times (1.01)^n = (1.01)^n \quad (n = 0, 1, 2, \ldots).$$

The 10th term is

$$x_9 = 1.01^9 = 1.094 \text{ (to 4 s.f.)}.$$

(b) Since $a = 10$ and $r = -1.5$, the alternative closed form is

$$y_n = 10(-1.5)^n \quad (n = 0, 1, 2, \ldots).$$

The 10th term is

$$y_9 = 10(-1.5)^9 = -384.4 \text{ (to 4 s.f.)}.$$

Solution 4.1

(a) This is a linear recurrence sequence, with parameters $a = 1$, $r = 2$ and $d = 1$.

(b) This is a linear recurrence sequence, with parameters $a = 1$, $r = -1$ and $d = -1$.

(c) This is a linear recurrence sequence, with parameters $a = 2$, $r = 1$ and $d = -1.5$. (Since $r = 1$, this sequence is also arithmetic.)

Solution 4.2

(a) The first term is $a = 3$. To find r and d, we follow the approach in Frame 1, using the terms $x_1 = 3$, $x_2 = 5$ and $x_3 = 9$. We obtain

$$5 = 3r + d, \tag{1}$$
$$9 = 5r + d. \tag{2}$$

Subtracting equation (1) from equation (2) gives

$$2r = 4;$$

that is, $r = 2$. Then, from equation (1), $d = 5 - 6 = -1$.

So the recurrence system is

$$x_1 = 3, \quad x_{n+1} = 2x_n - 1 \quad (n = 1, 2, 3, \ldots).$$

This gives

$$\begin{aligned} x_4 &= 2x_3 - 1 \\ &= 2 \times 9 - 1 = 17, \end{aligned}$$

as expected.

The next two terms are

$$x_5 = 2x_4 - 1 = 33,$$
$$x_6 = 2x_5 - 1 = 65.$$

The graph is as follows.

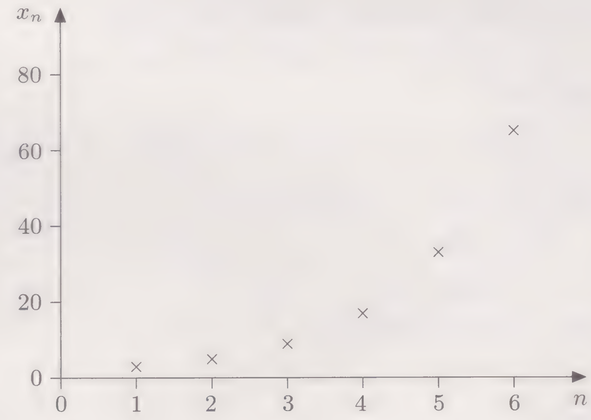


Figure S.10

(b) The first term is $a = 12$. To find r and d, we use the terms $x_1 = 12$, $x_2 = -3$ and $x_3 = 2$. We obtain

$$-3 = 12r + d, \tag{1}$$
$$2 = -3r + d. \tag{2}$$

Subtracting equation (2) from equation (1) gives

$$15r = -5;$$

that is, $r = -1/3$. Then, from equation (1),

$$d = -3 - 12 \times (-\tfrac{1}{3}) = 1.$$

So the recurrence system is

$$x_1 = 12, \quad x_{n+1} = -\tfrac{1}{3}x_n + 1 \quad (n = 1, 2, 3, \ldots).$$

This gives

$$\begin{aligned} x_4 &= -\tfrac{1}{3}x_3 + 1 \\ &= -\tfrac{1}{3} \times 2 + 1 = \tfrac{1}{3}, \end{aligned}$$

as expected.

The next two terms are

$$\begin{aligned} x_5 &= -\tfrac{1}{3}x_4 + 1 = -\tfrac{1}{3} \times \tfrac{1}{3} + 1 = \tfrac{8}{9}, \\ x_6 &= -\tfrac{1}{3}x_5 + 1 = -\tfrac{1}{3} \times \tfrac{8}{9} + 1 = \tfrac{19}{27}. \end{aligned}$$

The graph is as follows.

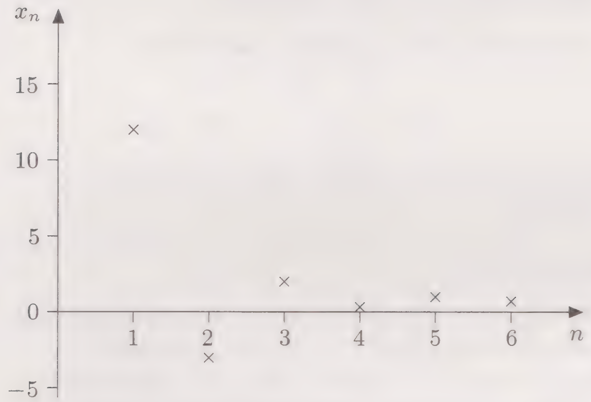


Figure S.11

(c) The first term is $a = 0$. To find r and d, we use the terms $x_1 = 0$, $x_2 = 2$ and $x_3 = 0$. We obtain

$$2 = d, \qquad (1)$$
$$0 = 2r + d. \qquad (2)$$

Now equation (1) gives $d = 2$ and then equation (2) gives $r = -1$.

So the recurrence system is

$$x_1 = 0, \quad x_{n+1} = -x_n + 2 \quad (n = 1, 2, 3, \ldots).$$

This gives

$$\begin{aligned} x_4 &= -x_3 + 2 \\ &= -0 + 2 = 2, \end{aligned}$$

as expected.

The next two terms are

$$\begin{aligned} x_5 &= -x_4 + 2 = -2 + 2 = 0, \\ x_6 &= -x_5 + 2 = -0 + 2 = 2. \end{aligned}$$

The graph is as follows.

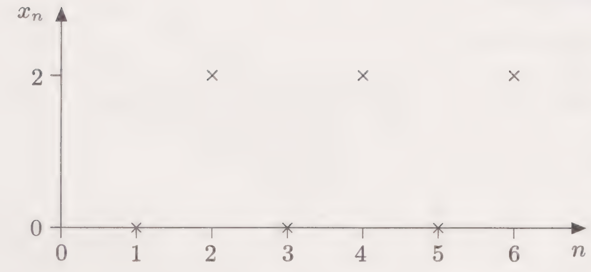


Figure S.12

Solution 4.3

For ease of reference, the closed form is given here:

$$x_n = \left(a + \frac{d}{r-1}\right) r^{n-1} - \frac{d}{r-1} \quad (n = 1, 2, 3, \ldots).$$

(a) Since $a = 3$, $r = 2$ and $d = -1$, we have

$$\frac{d}{r-1} = \frac{-1}{1} = -1,$$

and so the closed form is

$$\begin{aligned} x_n &= (3 - 1) \times 2^{n-1} - (-1) \\ &= 2^n + 1 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

Hence

$$x_{10} = 2^{10} + 1 = 1025.$$

(b) Since $a = 12$, $r = -1/3$ and $d = 1$, we have

$$\frac{d}{r-1} = \frac{1}{-1/3 - 1} = \frac{1}{-4/3} = -\tfrac{3}{4},$$

and so the closed form is

$$\begin{aligned} x_n &= \left(12 - \tfrac{3}{4}\right) \times (-\tfrac{1}{3})^{n-1} - (-\tfrac{3}{4}) \\ &= \tfrac{45}{4}(-\tfrac{1}{3})^{n-1} + \tfrac{3}{4} \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

Hence

$$x_{10} = \tfrac{45}{4}(-\tfrac{1}{3})^9 + \tfrac{3}{4} = 0.749 \text{ (to 3 d.p.)}.$$

(c) Since $a = 0$, $r = -1$ and $d = 2$, we have

$$\frac{d}{r-1} = \frac{2}{-2} = -1,$$

and so the closed form is

$$\begin{aligned} x_n &= (0 - 1) \times (-1)^{n-1} - (-1) \\ &= (-1)^n + 1 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

Hence

$$x_{10} = (-1)^{10} + 1 = 1 + 1 = 2.$$

Solution 4.4

The closed form

$$x_n = \left(a + \frac{d}{r-1}\right) r^{n-1} - \frac{d}{r-1} \quad (n = 1, 2, 3, \ldots)$$

gives

$$x_1 = \left(a + \frac{d}{r-1}\right) r^{0} - \frac{d}{r-1}.$$

Since $r^0 = 1$, this reduces to a, which is indeed the first term. Also

$$\begin{aligned} x_2 &= \left(a + \frac{d}{r-1}\right) r^{1} - \frac{d}{r-1} \\ &= ar + \frac{rd-d}{r-1} \\ &= ar + \frac{d(r-1)}{r-1} \\ &= ar + d, \end{aligned}$$

which is the second term.

Solution 4.5

(a) Since $a = 50$, $r = 0.9$ and $d = 8$, we have

$$\frac{d}{r-1} = \frac{8}{-0.1} = -80,$$

and so the closed form is

$$\begin{aligned} x_n &= (50 - 80)(0.9)^n - (-80) \\ &= -30(0.9)^n + 80 \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

Hence the 10th term is

$$x_9 = -30(0.9)^9 + 80 = 68.377 \text{ (to 3 d.p.)}.$$

(b) Since $a = -1$, $r = -2$ and $d = 1$, we have

$$\frac{d}{r-1} = \frac{1}{-2-1} = -\tfrac{1}{3},$$

and so the closed form is

$$\begin{aligned} y_n &= (-1 - \tfrac{1}{3})(-2)^n - (-\tfrac{1}{3}) \\ &= -\tfrac{4}{3}(-2)^n + \tfrac{1}{3} \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

Hence the 10th term is

$$y_9 = -\tfrac{4}{3}(-2)^9 + \tfrac{1}{3} = -\tfrac{4}{3}(-512) + \tfrac{1}{3} = 683.$$

(c) Since $r = 1$, this is in fact an *arithmetic* sequence, and so, since $a = 1$ and $d = 1/2$, the closed form is

$$z_n = 1 + n(\tfrac{1}{2}) = 1 + \frac{n}{2} \quad (n = 0, 1, 2, \ldots).$$

Hence the 10th term is

$$z_9 = 1 + \tfrac{9}{2} = 5.5.$$

Solution 5.1

(a) The closed form is

$$x_n = 2^n + 1 \quad (n = 1, 2, 3, \ldots).$$

As n becomes large, 2^n becomes arbitrarily large (since $2 > 1$), and so therefore does x_n.

(In symbols, $x_n \to \infty$ as $n \to \infty$.)

(b) The closed form is

$$x_n = \tfrac{45}{4}(-\tfrac{1}{3})^{n-1} + \tfrac{3}{4} \quad (n = 1, 2, 3, \ldots).$$

As n becomes large, $\left(-\tfrac{1}{3}\right)^{n-1}$ becomes arbitrarily small but alternates in sign (since $-1 < -\tfrac{1}{3} < 0$). Hence

$$\tfrac{45}{4}\left(-\tfrac{1}{3}\right)^{n-1}$$

also becomes arbitrarily small and alternates in sign. Thus x_n becomes arbitrarily close to $\tfrac{3}{4}$, with its terms alternately above and below this value.

(In symbols, $x_n \to \tfrac{3}{4}$ as $n \to \infty$.)

(c) In this case, the sequence x_n alternates between 0 and 2. The closed form is

$$x_n = (-1)^n + 1 \quad (n = 1, 2, 3, \ldots),$$

but this tells us nothing extra: there is no more to tell!

Solution 7.1

The oil sequence, the bouncing ball sequence and the deer sequence all attempt to predict real-world behaviour and so are models. The other sequences were designed to specify particular events in the real world, so they are not models in this sense.

Solution 7.2

1. Predict the height of a bouncing ball after a given number of bounces. Collect data on heights to which a real ball rebounds.
2. Set up the geometric sequence h_n discussed in Section 3.
3. Construct a table of values of h_n.
4. Decide which values in the table are large enough to compare with the heights of a real bouncing ball.
5. Determine whether these values in the table form a 'good enough' prediction of the heights of a real bouncing ball. If not, try to explain why the comparison is poor, in order to construct a better model.

Solutions to Exercises

Solution 1.1

(a) $a_1 = 2^1 - 1^2 = 1,$
$a_2 = 2^2 - 2^2 = 0,$
$a_3 = 2^3 - 3^2 = -1,$
$a_4 = 2^4 - 4^2 = 0,$
$a_5 = 2^5 - 5^2 = 7,$
$a_{10} = 2^{10} - 10^2 = 924.$

(b) $b_1 = 1/1^2 = 1,$
$b_2 = 1/2^2 = 1/4,$
$b_3 = 1/3^2 = 1/9,$
$b_4 = 1/4^2 = 1/16,$
$b_5 = 1/5^2 = 1/25,$
$b_{10} = 1/10^2 = 1/100.$

(c) $c_0 = \sqrt{0} = 0,$
$c_1 = \sqrt{1} = 1,$
$c_2 = \sqrt{2} = 1.414$ (to 3 d.p.),
$c_3 = \sqrt{3} = 1.732$ (to 3 d.p.),
$c_4 = \sqrt{4} = 2,$
$c_9 = \sqrt{9} = 3.$

Solution 1.2

The graphs for the sequence a_n, b_n and c_n are as follows.

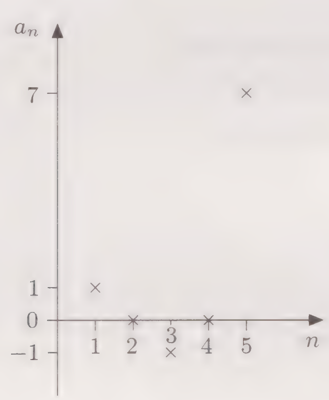


Figure S.13

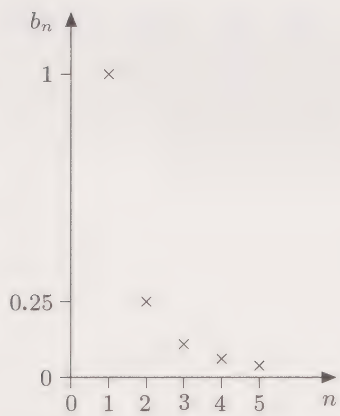


Figure S.14

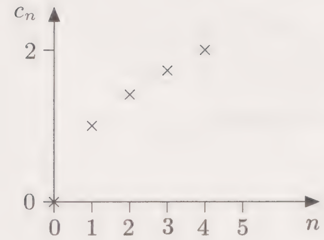


Figure S.15

Solution 1.3

(a) $a_1 = 1,$
$a_2 = 1/(a_1 + 1) = 1/(1 + 1) = 1/2,$
$a_3 = 1/(a_2 + 1) = 1/(1/2 + 1) = 1/(3/2) = 2/3,$
$a_4 = 1/(a_3 + 1) = 1/(2/3 + 1) = 1/(5/3) = 3/5,$
$a_5 = 1/(a_4 + 1) = 1/(3/5 + 1) = 1/(8/5) = 5/8.$

(b) $b_0 = 1,$
$b_1 = 2^{b_0} = 2^1 = 2,$
$b_2 = 2^{b_1} = 2^2 = 4,$
$b_3 = 2^{b_2} = 2^4 = 16,$
$b_4 = 2^{b_3} = 2^{16} = 65\,536.$

(c) $c_1 = 2,$
$c_2 = 1/c_1 = 1/2,$
$c_3 = 1/c_2 = 1/(1/2) = 2,$
$c_4 = 1/c_3 = 1/2,$
$c_5 = 1/c_4 = 1/(1/2) = 2.$

Solution 2.1

(a) (i) The first term is $a = 100$ and the difference is $d = 95 - 100 = -5$. So the recurrence system is

$$x_1 = 100, \quad x_{n+1} = x_n - 5 \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$x_5 = x_4 - 5 = 85 - 5 = 80,$$
$$x_6 = x_5 - 5 = 80 - 5 = 75.$$

The graph is as follows.

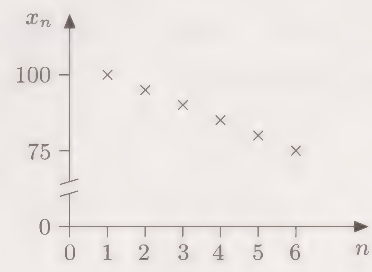


Figure S.16

(b) (i) The first term is $a = \frac{1}{4}$ and the difference is $d = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. So the recurrence system is

$$y_1 = \tfrac{1}{4}, \quad y_{n+1} = y_n + \tfrac{1}{4} \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$y_5 = y_4 + \tfrac{1}{4} = 1 + \tfrac{1}{4} = \tfrac{5}{4},$$
$$y_6 = y_5 + \tfrac{1}{4} = \tfrac{5}{4} + \tfrac{1}{4} = \tfrac{3}{2}.$$

The graph is as follows.

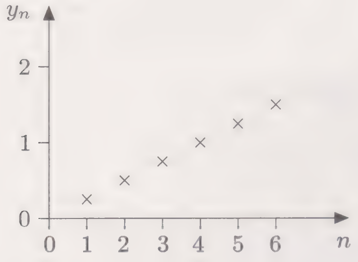


Figure S.17

Solution 2.2

(a) Since $a = 2$ and $d = -3$, the closed form is

$$\begin{aligned} x_n &= 2 - 3(n-1) \\ &= -3n + 5 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

The 10th term is

$$x_{10} = -3 \times 10 + 5 = -25.$$

(b) Since $a = 0$, $d = 0.2$ and the first term is y_0, the closed form is

$$y_n = 0 + 0.2n = 0.2n \quad (n = 0, 1, 2, \ldots).$$

The 10th term is

$$y_9 = 0.2 \times 9 = 1.8.$$

Solution 3.1

(a) (i) The first term is $a = 2$ and the ratio is $r = -3/2 = -1.5$. So the recurrence system is

$$x_1 = 2, \quad x_{n+1} = -1.5x_n \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$x_5 = -1.5x_4 = 10.125$$
$$x_6 = -1.5x_5 = -15.1875 = -15.188 \text{ (to 3 d.p.)}.$$

The graph is as follows.

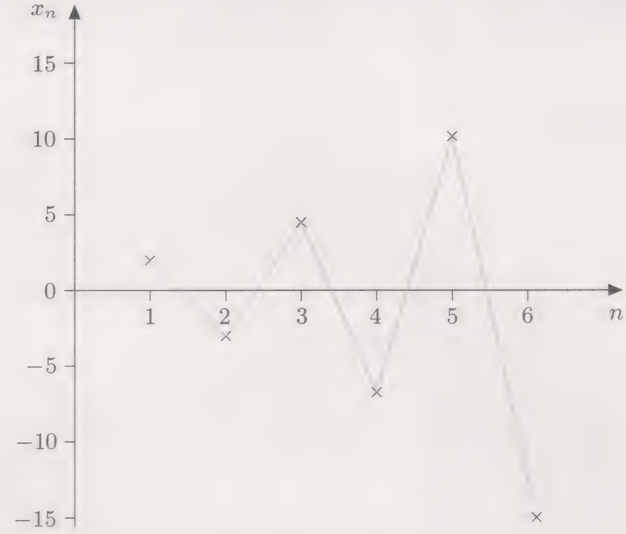


Figure S.18

(b) (i) The first term is $a = 100$ and the ratio is $r = 99/100 = 0.99$. So the recurrence system is

$$y_1 = 100, \quad y_{n+1} = 0.99y_n \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$\begin{aligned} y_5 &= 0.99y_4 \\ &= 0.99 \times 97.0299 \\ &= 96.059\,601 = 96.1 \text{ (to 3 s.f.)}, \end{aligned}$$

$$\begin{aligned} y_6 &= 0.99y_5 \\ &= 0.99 \times 96.059\,601 \\ &= 95.099\,005 = 95.1 \text{ (to 3 s.f.)}. \end{aligned}$$

The graph is as follows.

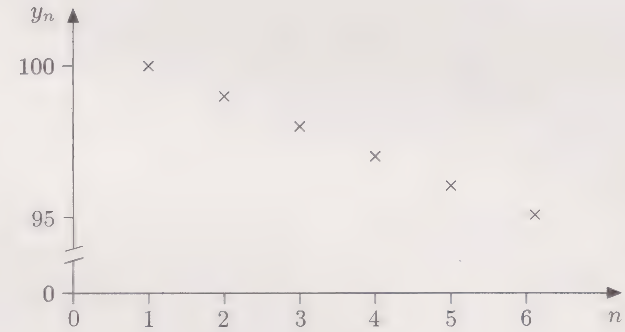


Figure S.19

Solution 3.2

(a) Since $a = 9$ and $r = -2/3$, the closed form is

$$x_n = 9\left(-\tfrac{2}{3}\right)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

The 10th term is

$$x_{10} = 9\left(-\tfrac{2}{3}\right)^{9} = -0.2341 \text{ (to 4 s.f.)}.$$

(b) Since $a = 1$, $r = 1.02$ and the first term is y_0, the closed form is

$$y_n = 1 \times 1.02^n = 1.02^n \quad (n = 0, 1, 2, \ldots).$$

The 10th term is

$$y_9 = 1.02^9 = 1.195 \text{ (to 4 s.f.)}.$$

Solution 4.1

(a) (i) The first term is $a = 1$. To find r and d, we use the terms $x_1 = 1$, $x_2 = 3.1$ and $x_3 = 5.41$. We obtain

$$3.1 = r + d, \tag{1}$$
$$5.41 = 3.1r + d. \tag{2}$$

Subtracting equation (1) from equation (2) gives

$$2.1r = 2.31;$$

that is, $r = 2.31/2.1 = 1.1$. Then, from equation (1),

$$d = 3.1 - r = 3.1 - 1.1 = 2.$$

So the recurrence system is

$$x_1 = 1, \quad x_{n+1} = 1.1x_n + 2 \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$\begin{aligned} x_5 &= 1.1x_4 + 2 \\ &= 1.1 \times 7.951 + 2 \\ &= 10.7461 = 10.746 \text{ (to 3 d.p.)}, \end{aligned}$$

$$\begin{aligned} x_6 &= 1.1x_5 + 2 \\ &= 1.1 \times 10.7461 + 2 \\ &= 13.82071 = 13.821 \text{ (to 3 d.p.)}. \end{aligned}$$

The graph is as follows.

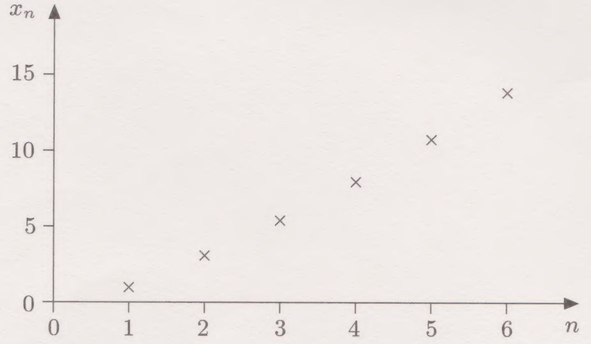


Figure S.20

(b) (i) The first term is $a = 4$. To find r and d, we use the terms $y_1 = 4$, $y_2 = 3$ and $y_3 = 2.75$. We obtain

$$3 = 4r + d, \tag{1}$$
$$2.75 = 3r + d. \tag{2}$$

Subtracting equation (2) from equation (1) gives

$$r = 0.25,$$

and so

$$d = 3 - 4r = 3 - 1 = 2.$$

So the recurrence system is

$$y_1 = 4, \quad y_{n+1} = 0.25y_n + 2 \quad (n = 1, 2, 3, \ldots).$$

(ii) The next two terms are

$$\begin{aligned} y_5 &= 0.25y_4 + 2 \\ &= 0.25 \times 2.6875 + 2 \\ &= 2.671875 = 2.672 \text{ (to 3 d.p.)}, \end{aligned}$$

$$\begin{aligned} y_6 &= 0.25y_5 + 2 \\ &= 0.25 \times 2.671875 + 2 \\ &= 2.66796875 = 2.668 \text{ (to 3 d.p.)}. \end{aligned}$$

The graph is as follows.

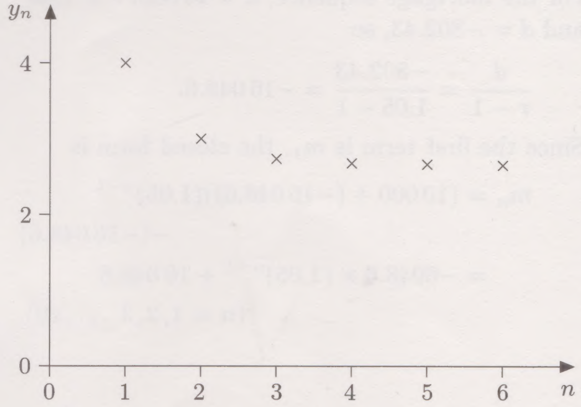


Figure S.21

Solution 4.2

(a) Since $a = 0$, $r = 3$ and $d = 1$, we have

$$\frac{d}{r-1} = \tfrac{1}{2},$$

and so the closed form is

$$\begin{aligned} x_n &= \left(0 + \tfrac{1}{2}\right) \times 3^{n-1} - \tfrac{1}{2} \\ &= \tfrac{1}{2} \times 3^{n-1} - \tfrac{1}{2} \\ &= \tfrac{1}{2}\left(3^{n-1} - 1\right) \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

Hence the 10th term is

$$x_{10} = \tfrac{1}{2}(3^9 - 1) = 9841.$$

(b) Since $a = 1$, $r = 1/2$ and $d = 1$, we have

$$\frac{d}{r-1} = \frac{1}{1/2 - 1} = -2,$$

Since the first term is y_0, the closed form is

$$\begin{aligned} y_n &= (1 + (-2)) \times (\tfrac{1}{2})^n - (-2) \\ &= -(\tfrac{1}{2})^n + 2 \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

Hence the 10th term is

$$y_9 = -(\tfrac{1}{2})^9 + 2 = 1.998 \text{ (to 3 d.p.)}.$$

(c) Since $r = 1$, z_n is an arithmetic sequence with $a = 100$, and $d = -0.1$. Since the first term is z_0, the closed form is

$$z_n = 100 - 0.1n \quad (n = 0, 1, 2, \ldots).$$

Hence the 10th term is

$$z_9 = 100 - 0.1 \times 9 = 99.1.$$

Solution 4.3

(a) For the deer population sequence, $a = 6000$, $r = 1.15$ and $d = -500$, so

$$\frac{d}{r-1} = \frac{-500}{1.15 - 1} = -3333.\dot{3}.$$

Since the first term is P_1, the closed form is

$$\begin{aligned} P_n &= (6000 + (-3333.\dot{3}))(1.15)^{n-1} - (-3333.\dot{3}) \\ &= 2666.\dot{6} \times (1.15)^{n-1} + 3333.\dot{3} \\ &\qquad (n = 1, 2, 3, \ldots). \end{aligned}$$

(b) For the mortgage sequence, $a = 10\,000$, $r = 1.05$ and $d = -802.43$, so

$$\frac{d}{r-1} = \frac{-802.43}{1.05 - 1} = -16\,048.6.$$

Since the first term is m_1, the closed form is

$$\begin{aligned} m_n &= (10\,000 + (-16\,048.6))(1.05)^{n-1} \\ &\qquad -(-16\,048.6) \\ &= -6048.6 \times (1.05)^{n-1} + 16\,048.6 \\ &\qquad (n = 1, 2, 3, \ldots, 20). \end{aligned}$$

Solution 5.1

(a) The closed form is

$$x_n = \tfrac{1}{2}(3^{n-1} - 1) \quad (n = 1, 2, 3, \ldots).$$

As n becomes large, 3^{n-1} becomes arbitrarily large (because $3 > 1$), and so therefore does x_n.

(In symbols, $x_n \to \infty$ as $n \to \infty$.)

(b) The closed form is

$$y_n = -(\tfrac{1}{2})^n + 2 \quad (n = 0, 1, 2, \ldots).$$

As n becomes large, $(\tfrac{1}{2})^n$ becomes arbitrarily small (since $0 < \tfrac{1}{2} < 1$) and so y_n approaches arbitrarily close to 2, with all its terms below this value.

(In symbols, $y_n \to 2$ as $n \to \infty$.)

(c) In this case, the closed form is

$$z_n = 100 - 0.1n \quad (n = 0, 1, 2, \ldots).$$

As n becomes large, $-0.1n$ becomes arbitrarily large and negative, and so therefore does z_n.

Solution 5.2

The closed form for the deer population sequence is

$$P_n = 2666.\dot{6} \times (1.15)^{n-1} + 3333.\dot{3} \quad (n = 1, 2, 3, \ldots).$$

As n becomes large, $(1.15)^{n-1}$ becomes arbitrarily large (since $1.15 > 1$) and so therefore does P_n.

(In symbols, $P_n \to \infty$ as $n \to \infty$.)

Note that since P_n becomes arbitrarily large, culling 500 deer each year will not control the population in the long term.

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